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Calabi-Yau Manifolds Over Finite Fields, I

Philip Candelas[1], Xenia de la Ossa[1]

and

Fernando Rodriguez Villegas[2]

[1] *Mathematical Institute*
Oxford University
24-29 St. Giles'
Oxford OX1 3LB, England

[2] *Department of Mathematics*
University of Texas
Austin
TX 78712, USA

ABSTRACT

We study Calabi–Yau manifolds defined over finite fields. These manifolds have parameters, which now also take values in the field and we compute the number of rational points of the manifold as a function of the parameters. The intriguing result is that it is possible to give explicit expressions for the number of rational points in terms of the periods of the holomorphic three-form. We show also, for a one parameter family of quintic threefolds, that the number of rational points of the manifold is closely related to as the number of rational points of the mirror manifold. Our interest is primarily with Calabi–Yau threefolds however we consider also the interesting case of elliptic curves and even the case of a quadric in $\mathbb{P}_1$ which is a zero dimensional Calabi–Yau manifold. This zero dimensional manifold has trivial dependence on the parameter over $\mathbb{C}$ but a not trivial arithmetic structure.

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1. Introduction

Calabi–Yau manifolds owe many remarkable properties to the special role that they play in relation to supersymmetry and to String Theory. It is a fundamental fact that these manifolds depend holomorphically on parameters that determine the complex structure and Kähler-class. The variation of the complex structure of a Calabi–Yau manifold, M, is naturally studied through it periods, that is the periods

$$\varpi_j \;=\; \int_{\Gamma^j} \Omega \tag{1.1}$$

of the (unique) holomorphic (3,0)-form over a basis of three dimensional homology cycles. The periods describe how Ω moves inside $H^3(M,\mathbb{Z})$ and play a fundamental role in the geometry of the parameter space [1,2] and hence also in mirror symmetry. Closely related periods play a crucial role also in Seiberg-Witten theory [3].

It has been known for some time that the periods encode also arithmetic properties of the underlying manifold [4,5]. Consider, for example, the one parameter family of quintic threefolds, M_ψ, defined by the vanishing of the polynomials

$$P(x,\psi) \;=\; \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1x_2x_3x_4x_5 \tag{1.2}$$

in $\mathbb{P}^4$. It is a consequence of mirror symmetry that the Yukawa coupling may be expanded in the form

$$y_{ttt} \;=\; 5 + \sum_{k=1}^{\infty} \frac{n_k k^3 q^k}{1-q^k} \quad , \quad q = e^{2\pi i t} \tag{1.3}$$

where the coordinate t and the Yukawa coupling can both be expressed in terms of certain periods ϖ_0 and ϖ_1

$$y_{ttt} \;=\; \left(\frac{5}{2\pi i}\right)^3 \frac{5\,\psi^2}{\varpi_0(\psi)^2\,(1-\psi^5)} \left(\frac{d\psi}{dt}\right)^3 \quad , \quad t \;=\; \frac{1}{2\pi i}\,\frac{\varpi_1(\psi)}{\varpi_0(\psi)}\;. \tag{1.4}$$

The essential point is that (1.4) is a combination of periods while (1.3) is a series with integer coefficients n_k that count the numbers of rational curves of M_ψ. It has been pointed out by Lian and Yau [5] that integral series enter at an earlier stage. For example, in order

to expand the Yukawa coupling as a q-series we have to invert the relation between ψ and t. Doing this we find the integral series

$$\frac{1}{(5\psi)^5} = q+154\,q^2+179139\,q^3+313195944\,q^4+657313805125\,q^5+1531113959577750\,q^6 \\ +3815672803541261385\,q^7+9970002717955633142112\,q^8+\dots\ . \tag{1.5}$$

It is shown in [5] that the coefficients in this and similar series are indeed all integers and in some examples (though not the series for the quintic) that these series are Thompson series related to the Monster Group. The fact that the coefficients are integers in the series (1.5) is the more remarkable since the coefficients that appear in the period ϖ_1 are *not* all integers. The fact that the construction of Calabi–Yau manifolds involves combinatoric if not number theoretic procedures is evident as soon as one attempts any reasonably general construction and is especially evident in the toric constructions and the expressions that Batyrev [6] (see also [7]) gives for the Hodge numbers in terms of the toric data.

We turn now to a statement of our main result which expresses the number of rational points of the quintic (1.2) over $\mathbb{F}_p$ in terms of the periods of Ω. The derivation of this expression will be explained in the following sections. To write out the expression we have to recall some facts pertaining to the periods. The periods satisfy a system of differential equations, the Picard-Fuchs equations, with respect to the parameters. Specifically, for the family (1.2) there are 204 periods, but among them there are four periods which we shall denote by ϖ_0, ϖ_1, ϖ_2 and ϖ_3, which coincide with the periods of the mirror manifold and satisfy the equation

$$\mathcal{L}\,\varpi_j \;=\; 0 \quad \text{with} \quad \mathcal{L} \;=\; \vartheta^4 - 5\lambda\prod_{i=1}^{4}(5\vartheta+i)\ , \tag{1.6}$$

where here and in the following it is convenient to take the parameter to be

$$\lambda \;=\; \frac{1}{(5\psi)^5}\ ,$$

and ϑ denotes the logarithmic derivative $\lambda\frac{d}{d\lambda}$. Consider now the behaviour of the periods in the neighborhood of $\lambda = 0$. The Picard-Fuchs equation (1.6) has all four of its indices equal to zero. Thus the solutions are asymptotically like 1, $\log\lambda$, $\log^2\lambda$ and $\log^3\lambda$. We denote by f_0 the solution that is analytic at $\lambda = 0$. In fact this solution is given by the series

$$f_0(\lambda) \;=\; \sum_{m=0}^{\infty}\frac{(5m)!}{(m!)^5}\lambda^m\ .$$

We can choose the four periods to be of the form

$$\begin{aligned}
\varpi_0(\lambda) &= f_0(\lambda) \\
\varpi_1(\lambda) &= f_0(\lambda)\,\log\lambda + f_1(\lambda) \\
\varpi_2(\lambda) &= f_0(\lambda)\,\log^2\lambda + 2f_1(\lambda)\,\log\lambda + f_2(\lambda) \\
\varpi_3(\lambda) &= f_0(\lambda)\,\log^3\lambda + 3f_1(\lambda)\,\log^2\lambda + 3f_2(\lambda)\,\log\lambda + f_3(\lambda)
\end{aligned} \tag{1.7}$$

where the $f_j(\lambda)$ are power series in λ.

Now these periods correspond to certain cycles Γ^j in (1.1) and so can be computed, in principle, by direct integration. The Picard-Fuchs equation (1.6) is then the differential equation satisfied by the four $\varpi_j(\lambda)$. For the case of Calabi–Yau manifolds that can be realised as hypersurfaces in toric varieties, which is a wide class with the quintic as the simplest example, the manifold can be associated with the Newton polyhedron, Δ, of the monomials that appear in the polynomial that defines the hypersurface. In these cases, given Δ, there is a purely combinatoric way of finding a differential system that the periods satisfy; this yields the GKZ system [8]. A fact which is not fully understood is that the GKZ system is often of higher order than the Picard-Fuchs system. So while it is true that the periods satisfy the differential system that one deduces from Δ there are also often additional solutions of this system that are not periods. These extra solutions are called semiperiods and their appearance is somewhat mysterious. It turns out that for the quintic there is a semiperiod and it plays a role in our expressions.

For the quintic the GKZ operator, $\mathcal{L}^\Delta$, is related to the Picard-Fuchs operator by

$$\mathcal{L}^\Delta \;=\; \vartheta\,\mathcal{L} \;=\; \vartheta^5 - \lambda\prod_{i=1}^{5}(5\vartheta + i)\ .$$

The first equality shows that the periods $\varpi_j(\lambda)$ satisfy the new equation and the second equality shows that the new operator has all five of the indices corresponding to $\lambda = 0$ equal to zero. The semiperiod is thus of the form

$$\varpi_4(\lambda) \;=\; f_0(\lambda)\,\log^4\lambda + 4\,f_1(\lambda)\,\log^3\lambda + 6\,f_2(\lambda)\,\log^2\lambda + 4\,f_3(\lambda)\,\log\lambda + f_4(\lambda)\ , \tag{1.8}$$

with the power series f_0, f_1, f_2, f_3 as in (1.7). We denote by $\mathbb{F}_p$ the finite field with p elements and for $\psi \in \mathbb{F}_p$, $p \neq 5$, we denote by $\nu(\psi)$ the number of rational points of M

$$\nu(\psi) \;=\; \#\{x \in \mathbb{F}_p^5 \mid P(x,\psi) = 0 \ \}\ .$$

We denote also by ${}^n f_j$ the truncation of the series f_j to $n+1$ terms. Thus for example

$$ {}^n f_0(\lambda) \;=\; \sum_{m=0}^{n} \frac{(5m)!}{(m!)^5}\,\lambda^m \ . $$

With these conventions we can state our result most simply for the case that 5 does not divide $p-1$

$$\begin{aligned} \nu(\psi) \;=\; & {}^{(p-1)}f_0(\lambda^{p^4}) + \left(\frac{p}{1-p}\right)\ {}^{(p-1)}f_1'(\lambda^{p^4}) + \frac{1}{2!}\left(\frac{p}{1-p}\right)^2\ {}^{(p-1)}f_2''(\lambda^{p^4}) \\ & + \frac{1}{3!}\left(\frac{p}{1-p}\right)^3\ {}^{(p-1)}f_3'''(\lambda^{p^4}) + \frac{1}{4!}\left(\frac{p}{1-p}\right)^4\ {}^{(p-1)}f_4''''(\lambda^{p^4}) \qquad (\mathrm{mod}\, p^5)\ , \end{aligned} \tag{1.9}$$

where the coefficients $\frac{1}{j!}\left(\frac{p}{1-p}\right)^j$ are understood to be expanded p-adically. We extend this result in three directions: by writing an exact p-adic expression, by extending the result to finer fields with $q=p^s$ elements and by extending the result to cover the interesting case that $5|q-1$. If $5\not| q-1$ then the generalization is immediate. We are simply to replace p by q, extend the sum to infinity with a certain natural definition of the new terms f_j for $j>4$ and replace λ by its Teichmüller representative. The case that $5|q-1$ is of practical importance since for every $p\neq 5$ there is a least integer ρ such that $5|p^\rho-1$ and, depending on p, $\rho=1,2$ or 4. Moreover this case corresponds to the existence of nontrivial fifth roots of unity and is of additional interest in virtue of mirror symmetry. Precisely when $5|q-1$ there are additional contributions to $\nu(\psi)$ arising from the other 200 periods. We compute these contributions which, apart from corresponding to these other periods, have a structure analogous to the terms we have exhibited.

There is an intuitive reason that $\nu(\psi)$ should be related to the periods in virtue of the action of the Frobenius map on the cohomology of M. We may regard the quintic as a polynomial with variables x_i that take values in some sufficiently large field $K\supset\mathbb{F}_p$ but with coefficients in $\mathbb{F}_p$. In this case the Frobenius map

$$(x_1,x_2,x_3,x_4,x_5)\mapsto(x_1^p,x_2^p,x_3^p,x_4^p,x_5^p)$$

acts as an automorphism on M since

$$P(x,\psi)\equiv 0 \pmod p \quad\Leftrightarrow\quad 0\equiv P(x,\psi)^p\equiv P(x^p,\psi) \pmod p$$

this is an automorphism since if $x^p\equiv y^p$ then $x^p-y^p\equiv 0$ so $(x-y)^p\equiv 0$ and $x-y\equiv 0$. The fixed points of the Frobenius map are the points for which $x^p\equiv x$ which is precisely

the condition that characterises $\mathbb{F}_p \subset K$. Thus the rational points of M are precisely the fixed points of the Frobenius map. Their number is then the Euler number of the fixed point set and the Lefshetz theorem relates this to a trace over the Frobenius action on the cohomology of M. In effect what we do in this paper is to evaluate this trace though we do so somewhat indirectly.

This paper is principally concerned with the calculation of the number of rational points of the quintic over $\mathbb{F}_q$. We have attempted to make the article reasonably self-contained and so we include in §2 a number of preliminaries relating to finite fields and p-adic analysis which, though standard, are not widely known to physicists. In §3 we give an account of the periods and a class of semiperiods which we define and which prove to be of importance for the computation of $\nu(\psi)$. In §4 we present the zeroth order calculation, that is the calculation of $\nu(\psi)$ mod p, and we derive the result

$$\nu(\psi) \;\equiv\; {}^{[p/5]}f_0(\lambda) \mod p$$

where $[p/5]$ denotes the integer part of $p/5$. This generalizes in a straightforward way a result that is well known for elliptic curves. It was this result that aroused our interest in this calculation since one inevitably asks why $\nu(\psi)$ should be related to a period.

It is perhaps natural to expand the number of points in base p

$$\nu(\psi) \;=\; \nu_0(\psi) + p\,\nu_1(\psi) + p^2\,\nu_2(\psi) + p^3\,\nu_3(\psi) + p^4\,\nu_4(\psi),$$

with $0 \le \nu_i(\psi) \le p-1$, and to seek to evaluate the coefficients successively by performing perturbation theory in powers of p. This leads inevitably to p-adic analysis since it is in this sense that the successive terms are smaller and smaller. Whereas the zeroth order calculation is elementary the first order calculation, neglecting terms of $\mathcal{O}\big(p^2\big)$, requires more serious computation at least computation that is more serious for the untrained physicist. The answer at first order is

$$\begin{aligned}\nu(\psi) = \;& {}^{(p-1)}f_0(\lambda^p) + p\;\,{}^{(p-1)}f_1'(\lambda^p) \\ & - \delta_p\, p \sum_{\mathbf{v}} \frac{\gamma_{\mathbf{v}}}{\prod_{i=1}^5 (v_i k)!}\;\,{}^{(p-1)}{}_2F_1\big(a_{\mathbf{v}}, b_{\mathbf{v}}; c_{\mathbf{v}};\, \psi^{-5}\big) + \mathcal{O}\big(p^2\big)\end{aligned}$$

where $\delta_p = 1$ if $5|p-1$ and $\delta_p = 0$ otherwise, $k = (p-1)/5$ and the $\mathbf{v}$-sum is over quintic monomials $x^{\mathbf{v}} = x_1^{v_1} \ldots x_5^{v_5}$ with certain coefficients $\gamma_{\mathbf{v}}$ that account for multiplicity. The ${}_2F_1$'s are the other 200 periods and it is at this order that we first see their contribution. Since the first order contribution was difficult to calculate and the result seems to show

a pattern it is natural to proceed by making an ansatz and checking this by counting the number of solutions to $P(x,\psi)=0$ with a computer for low values of p. This process leads to the expression (1.9) given above. In §6 we take this expression and extend it to an exact p-adic expression. In effect we extend the the sum in (1.9) to infinity and drop the proviso mod p^5. We are then able to perform the sum over j to obtain the expression in the form

$$\nu(\psi) \;=\; \sum_{m=0}^{q-1} \beta_{s,m} \mathrm{Teich}^m(\lambda)$$

in terms of the Teichmüller representative of λ and coefficients $\beta_{s,m}$ for which we give exact p-adic expressions in terms of the p-adic Γ-function. It seems that these expressions are interesting also for Calabi–Yau manifolds of lower dimension and we discuss this in §7 for a Calabi–Yau manifold of dimension zero, a quadric in $\mathbb{P}_1$,

$$x_1^2 + x_2^2 - 2\psi\, x_1 x_2 \;=\; 0\ .$$

Over $\mathbb{C}$ there are of course two solutions for $\psi^2 \neq 1$ and one solution when $\psi^2 = 1$. Over $\mathbb{PF}_p$ there can be 0, 1 or 2 solutions depending on ψ and p. It is a standard exercise in number theory to calculate the number of solutions and this number is usually expressed in terms of the Legendre symbol. We find it interesting that it is also possible to define periods and semiperiods for this situation and calculate the number of periods in terms of these. This leads to interesting relations between the periods and the Legendre symbol.

In §8 we turn to the case of elliptic curves. For the case of a cubic in $\mathbb{P}_2$ one may simply replace 5 by 3 in our expressions for the quintic to obtain expressions for the number of rational points. For a nonsingular elliptic curve we do not have to consider separately the case $3|q-1$ since there are no contributions from 'other periods'. A fact that is interesting is that the number of rational points for the manifold and for its mirror are the same.

In §9 we turn to a direct evaluation of $\nu(\psi)$ in terms of Dwork's character and Gauss sums. This establishes the expressions that were given previously. It is perhaps possible to omit the previous sections and begin the paper here. The evaluation of $\nu(\psi)$ in terms of Gauss sums, however, obscures the relation to the periods that we have sought to stress. It also requires a knowledge of the Gross-Koblitz formula which, though standard, is established by appeal to sophisticated cohomology theories. We derive here the complete form of the contributions of the other 200 periods whose form we new previously only mod p^2. These contributions are in correspondence with monomials of degree 5 and degree 10. The fact that monomials of degree contribute is surprising. The monomials in question

are those of the form $x_1^4\,x_2^3\,x_3^2\,x_4$ together with their permutations, and these contribute precisely when $\psi^5 = 1$ which is the case that the manifold singularises to a conifold.

A companion paper [9] deals with mirror symmetry and the application of our expressions to the zeta-function. Mirror symmetry impinges on the discussion since, apart from other considerations, the contribution of the degree 5 monomials to the count of rational points for the quintic has an appealing interpretation in terms of the monomial divisor mirror map whereby it is natural to associate these contributions with the rational points of the exceptional divisors that are introduced as blow ups of the mirror manifold in the toric construction.

2. **Preliminaries**

2.1. Field theory for physicists

A (number) field $\mathbb{F}$ is a set on which it is possible to perform the operations of arithmetic. That is the operations of addition and multiplication are defined and have the usual associative and distributive properties. Moreover $\mathbb{F}$ is an abelian group with respect to addition and $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$ is an abelian group with respect to multiplication. A field can have a finite or an infinite number of elements. The rationals, $\mathbb{Q}$, the reals, $\mathbb{R}$, and the complex numbers, $\mathbb{C}$, are fields but the integers, $\mathbb{Z}$, do not form a field since apart from ± 1 the inverse of an integer is not an integer.

Finite fields are classified up to isomorphism by the number of elements which is p^s for some prime p and integer s. There is therefore no ambiguity in writing $\mathbb{F}_{p^s}$ to denote such a field. Moreover these fields exist for each p and s.

The simplest finite field is the set of integers mod p

$$\mathbb{F}_p \;=\; \{0, 1, \ldots, p-1\}$$

and the field $\mathbb{F}_{p^s}$ is the field obtained by adjoining to $\mathbb{F}_p$ the root of an irreducible monic polynomial that has coefficients in $\mathbb{F}_p$ and is of order s. If ρ denotes a root of such a polynomial then $\mathbb{F}_{p^s}$ is the set

$$\mathbb{F}_{p^s} \;=\; \left\{ a_0 + a_1\rho + a_2\rho^2 + \ldots + a_{s-1}\rho^{s-1} \mid a_i \in \mathbb{F}_p \right\}$$

and it does not matter which irreducible polynomial or which root is chosen.

It is important that p should be prime. If we were to consider integers mod 6, say, we would have $2 \times 3 \equiv 0$ so that neither 2 nor 3 could have an inverse. If however p is prime then every nonzero element of the field has an inverse as we illustrate below for the case of $\mathbb{F}_7$. A fact that is not obvious is that $\mathbb{F}_p^*$ and more generally $\mathbb{F}_{p^s}^*$ are cyclic as multiplicative groups. In other words for each such group there exists a generator g such that for every $x \in \mathbb{F}_{p^s}^*$ we can write $x = g^j$ for some j in the range $0 \le j \le p^s - 2$. For the case of $\mathbb{F}_7$ the generator can be taken to be 3 as we illustrate in the table.

We shall be concerned later with the existence of nontrivial fifth roots of unity. That is when do there exist elements $\zeta \in \mathbb{F}_{p^s}^*$, $\zeta \neq 1$ such that $\zeta^5 = 1$. Since $\mathbb{F}_{p^s}^*$ is a group of

order $p^s - 1$ and a nontrivial fifth root, when it exists, generates a subgroup of order 5 a nontrivial fifth root cannot exist unless $5|p^s - 1$. Since the group $\mathbb{F}_{p^s}^*$ is in fact cyclic the ζ's exist precisely when the order of the group is divisible by 5 and the nontrivial fifth roots of unity are then $\zeta = g^{rk}$ with $r = 1, \ldots, 4$ and $k = (p^s - 1)/5$.

$\mathbb{F}_7$							
x	0	1	2	3	4	5	6
x^{-1}	*	1	4	5	2	3	6
$y\ ;\ 3^y = x$	*	0	2	1	4	5	3

There are two elementary facts that will be very useful to us. The first, an old result going back to Fermat, is that $a^p = a$, or equivalently $a(a^{p-1} - 1) = 0$. Thus

$$a^{p-1} \ = \ \begin{cases} 1, \text{ if } a \neq 0 \\ \\ 0, \text{ if } a = 0 \ . \end{cases} \tag{2.1}$$

We will also need to evaluate sums of powers of field elements. Consider therefore the sum

$$\sum_{a \in \mathbb{F}_p} a^n \ = \ \sum_{a \in \mathbb{F}_p} (ba)^n \ = \ b^n \sum_{a \in \mathbb{F}_p} a^n \ \ ; \ \ b \neq 0$$

Hence we have

$$\sum_{a \in \mathbb{F}_p} a^n \ = \ \begin{cases} 0, \text{ if } p-1 \text{ does not divide } n \\ \\ -1, \text{ if } p-1 \text{ divides } n \ . \end{cases} \tag{2.2}$$

For a given prime p the number of rational points, that is solutions of the equation $P(x, \psi) = 0$, is a definite integer which is moreover less than p^5 since there are p^5 points altogether in $\mathbb{F}_p^5$.

It is natural to expand the number of solutions in base p and write

$$\nu \ = \ \nu^{(0)} + \nu^{(1)}\, p + \nu^{(2)}\, p^2 + \nu^{(3)}\, p^3 + \nu^{(4)}\, p^4 \tag{2.3}$$

with $0 \leq \nu^{(j)} \leq p - 1$ and then to evaluate mod p^2, mod p^3, and so on. One proceeds by expanding all quantities as power series in p as if performing perturbation theory. This leads naturally into p-adic analysis since it is in this sense that it is legitimate to regard p as a small parameter.

2.2. p-adic numbers

Given an $r \in \mathbb{Q}$ and a prime p we can write

$$r \;=\; \frac{m}{n} \;=\; \frac{m_0}{n_0}\, p^{\alpha}$$

where m_0, n_0 and p have no common factor. The p-adic norm of r is defined to be

$$\|r\|_p \;=\; p^{-\alpha}\;, \quad \|0\|_p \;=\; 0$$

and is a norm; that is it has the properties

$$\begin{aligned} \|r\|_p &\geq 0, \quad \text{with equality if and only if } r = 0 \\ \|r_1\, r_2\|_p &= \|r_1\|_p\, \|r_2\|_p \\ \|r_1 + r_2\|_p &\leq \|r_1\|_p + \|r_2\|_p\;. \end{aligned}$$

With respect to the p-adic norm a number is smaller the more it is divisible by p. For the series (2.3), for example, the successive terms decrease with respect to the p-adic norm while they increase with respect to the usual norm. If this seems artificial then we refer the skeptical reader to Ostrowski's theorem [10] which states that any nontrivial norm is equivalent to the (usual) Archimedean norm or to the p-adic norm for some prime p.

Norms are important in analysis and enter into the way that we build up number systems. In the beginning there are integers and from these we obtain the rationals. Then we observe that there are sequences of rational numbers which converge, but to limits that are not rational, and we complete the rationals to the reals. Finally we extend the reals by adjoining $\sqrt{-1}$ to obtain the complex numbers; a field of the greatest utility for describing the natural world. Could we have deviated from this path? The answer is that the process of passing from $\mathbb{Q}$ to $\mathbb{R}$ required the notion of a limit and hence a norm. If instead of the usual Archimedian norm we use the p-adic norm we pass from $\mathbb{Q}$ to $\mathbb{Q}_p$, the field of p-adic numbers. By adjoining to $\mathbb{Q}_p$ the roots of all polynomials whose coefficients are in $\mathbb{Q}_p$ but whose roots are not in $\mathbb{Q}_p$ and then completing the result we arrive at $\mathbb{C}_p$ which is the p-adic analogue of $\mathbb{C}$. This construction has many applications in Number Theory. It is an intriguing question as to whether it has significant application in Physics.

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{C} \\ & & \downarrow & & & & \\ & & \mathbb{Q}_p & & & & \\ & & \downarrow & & & & \\ & & \mathbb{C}_p & & & & \end{array}$$

The p-adic numbers can be thought of as series of the form

$$X \;=\; \sum_{n=n_0}^{\infty} \xi_n \, p^n \;\;;\;\; \xi_n \in \mathbb{Z} \;,\;\; 0 \leq \xi_n \leq p-1 \;.$$

We shall make reference to the p-adic integers $\mathbb{Z}_p$ which is the ring of p-adic numbers for which $n_0 \geq 0$. Equivalently $\mathbb{Z}_p$ is the set of X's for which $\|X\|_p \leq 1$. If $n_0 = 0$ and $\xi_0 \neq 0$ then X is a *p-adic unit*. The p-adic units form a group, $\mathbb{Z}_p^*$. If $X \in \mathbb{Z}_p^*$ then both X and X^{-1} are p-adic integers and $\|X\|_p = 1$.

Since there is a norm defined for p-adic numbers there is also a p-adic analysis, with p-adic notions of limits and convergence. We will be concerned with the convergence of power series. Owing to the properties of the p-adic norm a p-adic power series $\sum_{n=0}^{\infty} \alpha_n x^n$ converges if and only if $\alpha_n x^n \to 0$ as $n \to \infty$. As in classical analysis power series have a radius of convergence but in p-adic analysis there is no notion of conditional convergence. If $\alpha_n x_0^n \to 0$ for some x_0 then $\alpha_n x^n \to 0$ for all x with $\|x\|_p \leq \|x_0\|_p$. In particular a power series either converges for all the points on the circumference of its disk of convergence or it diverges for all these points. Moreover the circumference of a disk has a 'thickness' in p-adic analysis. Consider the disk

$$D(a,R) \;=\; \left\{ x \in \mathbb{Q}_p \;\middle|\; \|x-a\|_p \leq R \right\}$$

and let b be a point such that $\|b-a\|_p = R$. Consider now the disk $D(b,r)$ for some $r < R$. If $x \in D(b,r)$ then

$$\|x-a\|_p \;=\; \|(x-b)+(b-a)\|_p \;\leq \max\left(\|(x-b)\|_p,\; R\right) \;=\; R \;.$$

so in fact $D(b,r) \subset D(a,R)$.

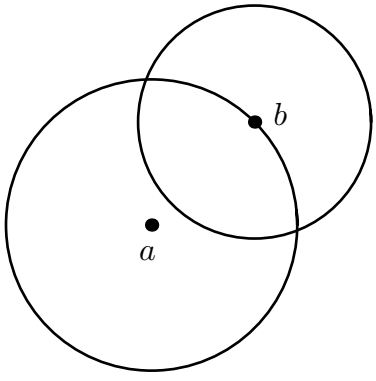

Figure 2.1: A naive sketch of the disks $D(a,R)$ and $D(b,r)$. In reality $D(b,r) \subset D(a,R)$.

In p-adic analysis the series expansion of a composite function $f(g(x))$ is more delicate than in classical analysis. Suppose f and g have series expansions that converge in disks D_f and D_g and let $D \subset D_g$ be the largest disk such that $x \in D \Rightarrow g(x) \in D_f$ then for $x \in D$ the series for $f(g(x))$ is obtained in the usual way by expanding f as a series in g and then g as a series in x. If the resulting series is $\sum_{n=0}^{\infty} \alpha_n x^n$ then this series converges to $f(g(x))$ for $x \in D$ however for $x \notin D$ the series may not converge or it may converge but not to $f(g(x))$. We will meet a textbook example of this phenomenon in §9. If $f(Y) = \exp(\pi Y)$ with exp denoting the usual exponential series and π a number in $\mathbb{C}_p$ such that $\pi^{p-1} = -p$ (this use of π has nothing to do with the classical number that goes by that name) then it can be shown that the largest disk for which the series expansion for Y converges in the disk $\|Y\|_p < 1$. If we take $g(X) = X - X^p$ then the X-disk that ensures $\|g(X)\|_p < 1$ is $\|X\|_p < 1$. For such X

$$f(g(X)) \;=\; \sum_{n=0}^{\infty} c_n \pi^n X^n$$

for certain coefficients c_n. It can be shown that the series on the right converges for $\|X\|_p < 1 + \epsilon$ for some fixed positive number ϵ so that the series $\sum_{n=0}^{\infty} c_n \pi^n$, obtained by setting $X = 1$, converges but it does not converge to $f(g(1)) = 1$.

2.3. Some useful congruences

We record here some useful congruences and relations which though elementary are perhaps not generally known to physicists.

The first is Wilson's Theorem

$$(p-1)! \;=\; -1 + \mathcal{O}(p) \;\; .$$

To see this think of $(p-1)!$ as $\prod_{x \in \mathbb{F}_p^*} x$. Every x has an inverse. If x and x^{-1} are distinct then they cancel in the product. We are left with x's that satisfy $x^2 - 1 = (x-1)(x+1) \equiv 0$ and these are 1 and $p-1$.

Next is a standard exercise [10, 11]: if $n \in \mathbb{Z}$ has the finite p-adic expansion, that is expansion in base p, $n = c_0 + c_1 p + \ldots + c_\ell p^\ell$ then the p-adic order of $n!$ is given by

$$\mathrm{ord}_p\,(n!) \;=\; \sum_{j=1}^{\ell} \left[\frac{n}{p^j}\right] \;=\; \frac{n - S(n)}{p-1} \tag{2.4}$$

where $S(n) = \sum_{j=0}^{\ell} c_j$ denotes the sum of digits function and [..] denotes the integer part. The first equality states that between 1 and n there are $\left[\frac{n}{p}\right]$ integers divisible by p, $\left[\frac{n}{p^2}\right]$ integers divisible by p^2 and so on. From this sum the result follows easily.

In this paper we shall need to make reference to the sums

$$\sigma_m \;=\; \sum_{j=1}^{m} \frac{1}{j} \quad \text{and more generally to} \quad \sigma_m^{(\ell)} \;=\; \sum_{j=1}^{m} \frac{1}{j^\ell} \tag{2.5}$$

σ_m being simply a shortened form of writing $\sigma_m^{(1)}$. Now it is follows from (2.2) that

$$\sigma_{p-1}^{(\ell)} \;=\; \mathcal{O}(p) \quad \text{if } (p-1) \nmid \ell \ .$$

a stronger result obtains for the important case $\ell = 1$ which is that

$$\sigma_{p-1} \;=\; \mathcal{O}\big(p^2\big) \quad \text{for } p > 3 \ .$$

To see this we evaluate $(p-1)!$ in two different ways

$$(p-1)! \;=\; \prod_{j=1}^{p-1}(p-j) \;=\; \prod_{j=1}^{p-1} j\left(1-\frac{p}{j}\right) \;=\; (p-1)!\left\{1 - p\sigma_{p-1} + \mathcal{O}\big(p^3\big)\right\}$$

from which the result follows.

Consider now the ratio

$$\frac{(rp)!}{((r-1)p)!} \;=\; rp\prod_{j=1}^{p-1}(rp-j) \;=\; r\,p!\prod_{j=1}^{p-1}\left(1-\frac{rp}{j}\right) \;=\; r\,p!\,\left\{1+\mathcal{O}\big(p^3\big)\right\} \ .$$

By multiplying ratios of this form we find

$$(ap)! \;=\; a!(p!)^a\left\{1+\mathcal{O}\big(p^3\big)\right\} \tag{2.6}$$

and from this it is easy to see that

$$(ap+b)! \;=\; (ap)!\prod_{j=1}^{b}(ap+j) \;=\; (ap)!b!\prod_{j=1}^{b}\left(1+\frac{ap}{j}\right) \;=\; a!b!(p!)^a\left\{1+ap\sigma_b+\mathcal{O}\big(p^2\big)\right\} \ .$$

Our expression for $(ap)!$ is valid for arbitrary positive integers a so we may replace a by ap to obtain

$$\big(ap^2\big)! \;=\; a!(p!)^{a(p+1)}\left\{1+\mathcal{O}\big(p^3\big)\right\} \ .$$

If in our expression for $(ap+b)!$ we replace a by $ap+b$ and b by c then we find

$$\big(ap^2+bp+c\big)! \;=\; a!b!c!(p!)^{a(p+1)+b}\left\{1+p(a\sigma_b+b\sigma_c)+\mathcal{O}\big(p^2\big)\right\} \ .$$

2.4. The p-adic Γ-Function

We recall here a number of relations concerning the p-adic Γ-function that we will need in the following. Recall [10, 11] that for $n \in \mathbb{Z}$, $n \geq 0$ this function is defined by

$$\Gamma_p(n+1) \;=\; (-1)^{n+1} {\prod_{j=1}^{n}}' j \quad , \qquad \Gamma_p(0) \;=\; 1 \; , \tag{2.7}$$

where the prime denotes that j's that are divisible by p are omitted from the product. The virtue of this omission as well as the improbable looking factor of $(-1)^{n+1}$ is that thus defined Γ_p is continuous and even analytic in the p-adic sense. Continuity in this sense means that $\Gamma_p(n + mp^N) \to \Gamma_p(n)$ as $N \to \infty$. Since Γ_p is continuous and the integers are dense in $\mathbb{Z}_p$ (the partial sums of the p-adic expansion of each $X \in \mathbb{Z}_p$ are integers that converge to X) the domain of Γ_p can be extended to $\mathbb{Z}_p$ by continuity. For $X \in \mathbb{Z}_p$ it then satisfies the recurrence

$$\frac{\Gamma_p(X+1)}{\Gamma_p(X)} \;=\; \begin{cases} -X \; ; & X \in \mathbb{Z}_p^* \\ -1 \;\; ; & X \in p\,\mathbb{Z}_p \; . \end{cases}$$

Now in computing the product ${\prod_{j=1}^{n}}' j$ the factors that are omitted are

$$p \times 2p \times 3p \times \cdots \times \left[\frac{n}{p}\right] p \;=\; \left[\frac{n}{p}\right]! \, p^{\left[\frac{n}{p}\right]} \; .$$

Hence

$$n! \;=\; (-1)^{n+1} \Gamma_p(n+1) p^{\left[\frac{n}{p}\right]} \left[\frac{n}{p}\right]! \; .$$

By writing $n = n_0 + n_1 p + \ldots n_\ell p^\ell$ in base p and iterating the above expression by using a similar expression to replace $[n/p]!$ and proceeding in this way we recover the useful textbook formula

$$n! \;=\; (-1)^{n+\ell+1} (-p)^{\frac{n-S(n)}{p-1}} \prod_{j=0}^{\ell} \Gamma_p\left(\left[\frac{n}{p^j}\right] + 1\right) \; . \tag{2.8}$$

Also useful is the reflection formula

$$\Gamma_p(X)\Gamma_p(1-X) \;=\; (-1)^{L(X)} \tag{2.9}$$

where $L(X)$ denotes the mod p reduction of the lowest digit of X to the range $\{1, 2, \ldots, p\}$, that is $L(X)$ is the lowest digit unless the lowest digit is 0 in which case $L(X) = p$. The previous formula is the analogue of the classical result

$$\Gamma(\xi)\Gamma(1-\xi) \;=\; \frac{\pi}{\sin(\pi\xi)} \;. \tag{2.10}$$

We shall have need also of the multiplication formula for Γ_p

$$\frac{\prod_{i=0}^{n-1} \Gamma_p\left(\frac{X+i}{n}\right)}{\Gamma_p(X)\prod_{i=1}^{n-1}\Gamma_p\left(\frac{i}{n}\right)} \;=\; n^{1-L(X)}\left(n^{-(p-1)}\right)^{X_1} \tag{2.11}$$

where n is an integer such that $p \not| \, n$ and X_1 is related to X by

$$X \;=\; L(X) + pX_1 \;.$$

Note that the right hand side of (2.11) has been carefully written so as to be defined for p-adic X. The quantity $n^{-(p-1)}$ is $1 + \mathcal{O}(p)$ and the result of elevating this to a p-adic power is defined via the binomial series. It is useful also to recall the classical multiplication formula for the Γ-function. This is usually presented in the form

$$\prod_{i=0}^{n-1} \Gamma\left(\frac{\xi+i}{n}\right) \;=\; (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-\xi}\,\Gamma(\xi) \;. \tag{2.12}$$

For our purposes it is better to eliminate the reference to the factors of π in this formula. By taking $\xi = 1$ we find

$$\prod_{i=1}^{n} \Gamma\left(\frac{i}{n}\right) \;=\; (2\pi)^{\frac{n-1}{2}} n^{-\frac{1}{2}}$$

and we use this expression to divide the previous one. This yields a formula remarkably similar in form to the p-adic expression

$$\frac{\prod_{i=0}^{n-1} \Gamma\left(\frac{\xi+i}{n}\right)}{\Gamma(\xi)\prod_{i=1}^{n-1}\Gamma\left(\frac{i}{n}\right)} \;=\; n^{1-\xi} \;. \tag{2.13}$$

2.5. The Teichmüller representative

Given an $a \in \mathbb{Z}$ such that $p \not| \, a$ we know that $a^{p-1} = 1 + \mathcal{O}(p)$ and by raising this relation to the power p we find that $a^{p(p-1)} = 1 + \mathcal{O}\big(p^2\big)$ and by continuing in this way we see that $a^{p^n(p-1)} = 1 + \mathcal{O}\big(p^{n+1}\big)$. In virtue of this it is natural to define the Teichmüller representative of a as the limit

$$\text{Teich}(a) \;=\; \lim_{n\to\infty} a^{p^n}$$

which exists as a p-adic number. The Teichmüller representative has the property that

$$\text{Teich}^p(a) \;=\; \text{Teich}(a)$$

exactly and so embeds $\mathbb{F}_p^*$ in $\mathbb{Q}_p^*$.

Put differently: if we think of an $a \in \mathbb{F}_p$ abstractly then $a^p = a$ exactly. If however we think of a as an integer then $a^p = a + \mathcal{O}(p)$. We will later be performing perturbation theory in powers of p and will want to work to $\mathcal{O}(p^n)$ for $n > 1$. It is therefore useful to observe that a^p also represents a in $\mathbb{F}_p$ (since $a^p = a + \mathcal{O}(p)$) however by the above $(a^p)^p = a^p + \mathcal{O}\big(p^2\big)$ so a^p is a better representative in this sense and the sequence a^{p^n}, which all represent a in $\mathbb{F}_p$, are successively better representatives. The Teichmüller representative is the limit of this process.

It is possible to define the Teichmüller representative also for fields $\mathbb{F}_q$ with $q = p^s$ elements. In this case

$$\text{Teich}(a) \;=\; \lim_{n\to\infty} a^{q^n} \;\; ; \quad a \in \mathbb{F}_q$$

and $\text{Teich}^q(a) = \text{Teich}(a)$ which embeds $\mathbb{F}_q^*$ in $\mathbb{Q}_p^*$.

2.6. The coefficients of the fundamental period

It is of interest to consider the p-adic structure of the coefficients $a_m = \frac{(5m)!}{(m!)^5}$. Since these numbers are multinomial coefficients they are all integers. Here are the first few:

1
120
113400
168168000
305540235000
623360743125120
1370874167589326400
317745907852341196800 0
765671445315319798183500 0
190106382026520307129782000 00
48334775757901219912115629238400
125285878026462826569986857692288000
32998183172842546530955925112303396000 0
88090418255500882369606044077538808320000 0
23788292796428186685570635582385374010240000 00

These coefficients have interesting structure if reduced mod p. For the case $p = 13$ the following is a list of the early coefficients thus reduced:

$$
p \left\{ \begin{array}{l}
\left[\frac{p}{5}\right]+1 \left\{ \begin{array}{l} 1,\,3,\,1,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\ 3,\,9,\,3,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\ 1,\,3,\,1,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \end{array} \right. \\
0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\
0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\
0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\
0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\
0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\
0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\
0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\
0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\
0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\
\left[\frac{p}{5}\right]+1 \left\{ \begin{array}{l} 3,\,9,\,3,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\ 9,\,1,\,9,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \\ \underbrace{3,\,9,\,3,}_{\left[\frac{p}{5}\right]+1}\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0, \end{array} \right.
\end{array} \right.
\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{p}
$$

In order to understand the block structure of the preceding table let us compute the p-adic order of the coefficients a_m. From (2.4) we see that this is given by

$$\mathrm{ord}_p\,(a_m) \;=\; \frac{5S(m)-S(5m)}{p-1}\ .$$

Let the p-adic expansions of m and $5m$ be

$$\begin{aligned} m &= c_0 + c_1 p + \ldots + c_r p^r \\ 5m &= d_0 + d_1 p + \ldots + d_s p^s\ . \end{aligned}$$

The digits of $5m$ are related to those of m by

$$\begin{aligned} 5c_0 &= d_0 + \tilde{d}_0 p \\ 5c_1 + \tilde{d}_0 &= d_1 + \tilde{d}_1 p \\ 5c_2 + \tilde{d}_1 &= d_2 + \tilde{d}_2 p \quad \text{etc.} \end{aligned}$$

where the $\tilde{d}_j$ are the 'carries'. We see that $5S(m)-S(5m) = (p-1)\sum_j \tilde{d}_j$ so that

$$\mathrm{ord}_p\,(a_m) \;=\; \sum_j \tilde{d}_j\ .$$

The right hand side of this expression is the total number of carries that are performed when we multiply m by 5. Consider again the list of the first p^2 coefficients a_m. We broke the list into lines of length p and then stacked these lines to form a column. It would be better to stack the lines upwards since then $a_{\xi+\eta p}$ would correspond to Cartesian coordinates (ξ,η). The coefficients with both ξ and η less than $p/5$ are $\mathcal{O}(1)$. If one of (ξ,η) is less than $p/5$ and the other is in the range $(p/5, 2p/5)$ then there is one carry and these coefficients are $\mathcal{O}(p)$. It is complicated to keep track of the precise number of carries but it is easy to see that $a_{\xi+\eta p}$ is at least as small as $\mathcal{O}\big(p^{[\xi/5]+[\eta/5]}\big)$. Thus the first p^2 coefficients form a square made up of rectangles of coefficients bounded the same order and the rectangles corresponding to the same order make up diagonal bands as in Figure 2.2. If we consider the first p^3 terms of the form $a_{\xi+\eta p+\zeta p^2}$ then we first assemble the terms into squares and then stack the squares into a cube. The small blocks then correspond to coefficients bounded by the same order as in Figure 2.3. The next stage is to stack cubes into a four-dimensional cube and so on.

Although we shall have to obtain more detailed information about the periods there are some very pretty results concerning the mod p reduction of the fundamental period that we pause to explain. Consider the coefficient a_m and write m as a multiple of p and

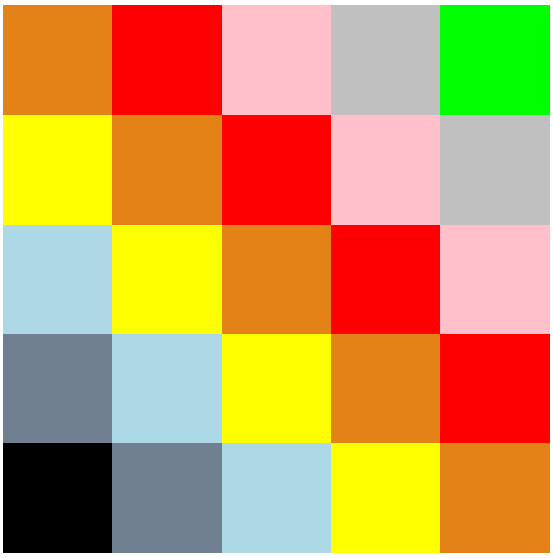

Figure 2.2: Bounds on the orders of the first p^2 coefficients a_m.

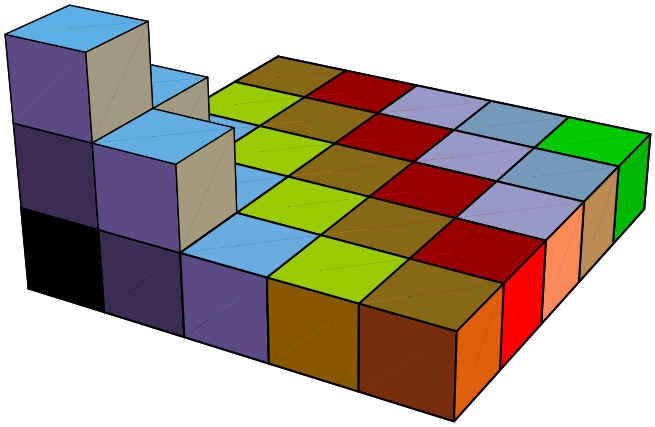

Figure 2.3: Bounds on the orders of the first p^3 coefficients a_m.

a remainder $m = rp + s$, $0 \leq s \leq p-1$. Now by means of (2.6) and the expression for $(ap+b)!$ it is easy to see that

$$a_{rp+s} \;=\; a_r a_s + \mathcal{O}(p) \;\; . \tag{2.14}$$

We use this in the expansion for the fundamental period

$$f_0(\lambda) \;\equiv\; \sum_{s=0}^{p-1}\sum_{r=0}^{\infty} a_{rp+s}\lambda^{rp+s} \;\equiv\; \sum_{s=0}^{\left[\frac{p}{5}\right]} a_s\lambda^s \sum_{r=0}^{\infty} a_r\lambda^{rp} \;\; .$$

Hence

$$\begin{aligned} f_0(\lambda) \;&\equiv\; {}^{[p/5]}f_0(\lambda)\, f_0(\lambda^p) \\ &\equiv\; {}^{[p/5]}f_0(\lambda)\;\; {}^{[p/5]}f_0(\lambda^p)\;\; {}^{[p/5]}f_0(\lambda^{p^2})\ldots \end{aligned} \tag{2.15}$$

The second congruence follows on iterating the first. The second congruence explains the mod p structure of $f_0(\lambda)$ and shows that to this order all the information is contained in ${}^{[p/5]}f_0$. We shall see later that to $\mathcal{O}\big(p^2\big)$ it is necessary to know only the first $[2p/5]$ terms and so on.

If we write

$${}^{[p/5]}f_0(\lambda) \;\equiv\; \frac{f_0(\lambda)}{f_0(\lambda^p)}$$

then we see also that the truncated period satisfies the Picard-Fuchs equation mod p

$$\mathcal{L}\left({}^{[p/5]}f_0(\lambda)\right) \;\equiv\; 0 \;, \quad \text{since} \quad \frac{d}{d\lambda}\,\lambda^p \;\equiv\; 0 \;\; .$$

3. The Periods and Semiperiods of the Quintic

3.1. The periods

In this section we will find all the periods and semiperiods of the one parameter family of quintic threefolds, M_ψ, defined by the vanishing of the polynomials

$$P(x,\psi)=\sum_{i=1}^{5}x_i^5-5\psi\ x_1x_2x_3x_4x_5$$

in $\mathbb{P}^4$. The nontrivial Hodge numbers of this manifold are $h^{1,1}=1$ and $h^{2,1}=101$. The dimension of the moduli space of the complex structure is $b_3=2(1+h^{2,1})=204$, which means there are 204 periods of the holomorphic $(3,0)$-form Ω. These periods satisfy a differential equation known as the Picard Fuchs equation which is of order 204. Fortunately, a task that seems quite impossible for a generic quintic three–fold, is greatly simplified by the fact the one parameter family of Calabi–Yau manifolds we are considering has a large group of automorphisms $\mathcal{A}=\mathcal{Z}\times\mathcal{P}$, where $\mathcal{P}$ is the group of permutations of the coordinates, and $\mathcal{Z}\cong\mathbb{Z}_5^3$ is the group of transformations of the form

$$(x_1,x_2,x_3,x_4,x_5)\ \longrightarrow\ (\zeta^{n_1}x_1,\zeta^{n_2}x_2,\zeta^{n_3}x_3,\zeta^{n_4}x_4,\zeta^{n_5}x_5)\ ;\quad \zeta^5=1\ ,\quad \sum_{i=1}^{5}n_i=0\ ,$$

generated by

$$\begin{aligned} g_1&=(4\ 1\ 0\ 0\ 0)\\ g_2&=(4\ 0\ 1\ 0\ 0)\\ g_3&=(4\ 0\ 0\ 1\ 0)\\ g_4&=(4\ 0\ 0\ 0\ 1)\ , \end{aligned}\tag{3.1}$$

where $g_1g_2g_3g_4=1$, and $g_1=(4\ 1\ 0\ 0\ 0)$, for example, means

$$g_1:(x_1,x_2,x_3,x_4,x_5)\ \longrightarrow\ (\zeta^4x_1,\zeta x_2,x_3,x_4,x_5)\ ;\quad \zeta^5=1\ .$$

There is a well defined prescription to find the Picard–Fuchs equation[12-16] for a manifold defined as a hypersurface in a toric variety. There is a one to one correspondence between the elements in the polynomial ring of the Calabi–Yau manifold and its periods, that is, for every monomial

$$x^{\mathbf{v}}=x_1^{v_1}x_2^{v_2}x_3^{v_3}x_4^{v_4}x_5^{v_5}$$

of degree $\sum_{i=1}^{5} v_i = 5w(\mathbf{v})$, $0 \leq w(\mathbf{v}) \leq 3$, in the polynomial ring of M_ψ, there exists a 3–cycle $\gamma_{\mathbf{v}} \in H_3(M_\psi, \mathbb{C})$ such that the corresponding period $\varpi_{\mathbf{v}}(\psi)$ is given by

$$\varpi_{\mathbf{v}}(\psi) = \int_{\gamma_{\mathbf{v}}} \Omega = \frac{1}{(2\pi i)^5} \int_{\Gamma} d^5x \ \frac{x^{\mathbf{v}}}{P^{w(\mathbf{v})+1}} \ , \tag{3.2}$$

where $\Gamma = C_1 \times C_2 \times \cdots \times C_5$ is a 5-torus that is the product of loops C_i that wind about the submanifolds $\partial_i P = 0$ in $\mathbb{C}^5$. Another way to say this is that there is a one to one correspondence between the cohomology classes in $H^3(M_\psi, \mathbb{C})$ and the equivalence classes of differential forms $d^5x \ x^{\mathbf{v}}/P^{w(\mathbf{v})+1}$ modulo exact forms.

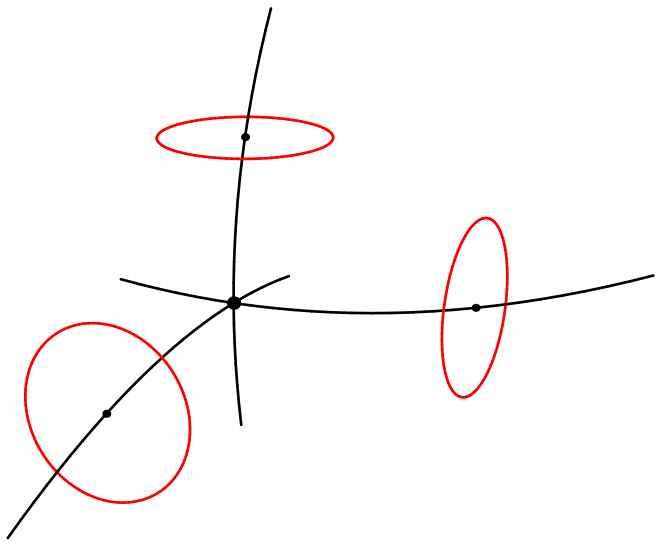

Figure 3.1: The contour Γ is the product of five S^1's that wind about the loci $\partial_i P = 0$. These loci are taken to have the origin as the unique point in which they all meet.

The Picard–Fuchs equation in our example, which could be a very complicated coupled system of differential equations, simplifies significantly because of the automorphisms of the manifold. The elements of the polynomial ring can be classified according to their transformation properties under the automorphisms, that is into representations of $\mathcal{A}$, and owing to the correspondence with the periods, the periods can also be classified accordingly. Consider now derivatives with respect to ψ of the periods (3.2). These derivatives produce

new periods which are in correspondence to monomials in the same representation of $\mathcal{A}$. For example, the period

$$\frac{\mathrm{d}}{\mathrm{d}\psi}\varpi_{\mathbf{v}}(\psi) = 5\big(w(\mathbf{v})+1\big)\frac{1}{(2\pi i)^5}\int_\Gamma d^5x\ \frac{x^{\mathbf{v+1}}}{P^{w(\mathbf{v})+2}}\ ,$$

corresponds to the monomial $x^{\mathbf{v+1}} = x^{\mathbf{v}}x_1x_2x_3x_4x_5$ of degree $5\big(w(\mathbf{v})+1\big)$, which is in the same representation as the monomial $x^{\mathbf{v}}$ because the monomial $x_1x_2x_3x_4x_5$ is invariant under the entire group of automorphisms. As a consequence, the Picard–Fuchs system of equations breaks down into independent blocks, one block for each representation of $\mathcal{A}$ in the polynomial ring.

The first step in finding the Picard–Fuchs equation is therefore to classify all the monomials in the polynomial ring according to their representation under the group of automorphisms. There are 126 monomials of degree 5, and they can be classified according to their transformation under $\mathcal{A}$ as follows:

case 1 :	6 invariant monomials	$\{x_1^5, x_2^5, x_3^5, x_4^5, x_5^5, x_1x_2x_3x_4x_5\}$
case 2 :	20 sets of 2 monomials	$\{x_1^4x_2, x_2^2x_3x_4x_5\}$, and permutations
case 3 :	20 sets of 1 monomial	$\{x_1^3x_2^2\}$, and permutations
case 4 :	30 sets of 1 monomial	$\{x_1^3x_2x_3\}$, and permutations
case 5 :	30 sets of 1 monomial	$\{x_1^2x_2^2x_3\}$, and permutations

Each incomplete representation of $\mathcal{A}$ above has to be completed to include also the monomials of degree zero, degree 10, and degree 15. There are a total of 101 representations and a differential operator corresponding to the periods in each representation, the order of each differential equation being the dimension of the representation. Note that not all the monomials in each representation are inequivalent. Two monomials are equivalent if they differ by an element in the Jacobian ideal, that is the ideal $\mathcal{I} = (\partial_i P)$ generated by the partial derivatives of P. For example, the monomials of degree 5 in the case 1 above, are all equivalent because $x_i^5 \simeq \psi Q$, and we choose $Q = x_1x_2x_3x_4x_5$ to be the representative. Similarly, the two monomials in each of the sets in the second case above are also equivalent since, for example, $x_1^4x_2 \simeq \psi\, x_2^2x_3x_4x_5$. The representations are then

case 1 :	1 dimension 4 representation	$\{1\ ,\ Q,\ Q^2,\ Q^3\}$,
case 2 :	20 dimension 2 representations	$\{x_1^4x_2,\ x_1^4x_2Q\}$, and permutations ,
case 3 :	20 dimension 2 representations	$\{x_1^3x_2^2,\ x_1^3x_2^2Q\}$, and permutations ,
case 4 :	30 dimension 2 representations	$\{x_1^3x_2x_3,\ x_1^3x_2x_3Q\}$, and permutations ,
case 5 :	30 dimension 2 representations	$\{x_1^2x_2^2x_3,\ x_1^2x_2^2x_3Q\}$, and permutations ,
case 6 :	120 dimension 1 representations	$\{x_1^4x_2^3x_3^2x_4\}$, and permutations .

The sixth case deserves some explanation. The monomial $x_1^4 x_2^3 x_3^2 x_4$ is in the ideal *except when* $\psi^5 = 1$, that is, except at the point in the moduli space where the manifold develops a conifold singularity. This can be seen by setting $\mu_i = x_i^4 x_{i+1}^3 x_{i+2}^2 x_{i+3}$, with the subscripts on the x_j understood mod 5. Since $x_i^4 \simeq x_{i+1}x_{i+2}x_{i+3}x_{i+4}$, modulo $\mathcal{I}$, we have $\mu_i \simeq \psi\,\mu_{i+1}$ and by repeating this relation we find

$$(1-\psi^5)\,\mu_i \;\simeq\; 0\ .$$

Equivalently, the monomial $x_1^4 x_2^3 x_3^2 x_4$ corresponds to a differential form which is exact except when $\psi^5 = 1$

$$-10\,(1-\psi^5)\,\frac{x_1^4 x_2^3 x_3^2 x_4}{P^3} \;=$$
$$\partial_1\left(\frac{x_2^3 x_3^2 x_4}{P^2}\right) + \partial_2\left(\frac{x_3^3 x_4^2 x_5}{P^2}\right) + \partial_3\left(\frac{x_1 x_4^3 x_5^2}{P^2}\right) + \partial_4\left(\frac{x_1^2 x_2 x_5^3}{P^2}\right) + \partial_5\left(\frac{x_1^3 x_2^2 x_3}{P^2}\right)\ .$$

Of course, as long as $\psi^5 \neq 1$, the period associated to this monomial is zero. The reason we are including this case will not become clear until later when we calculate the number of points over a finite field and we show that at $\psi^5 = 1$ there is a "period" associated to this monomial which gives an exceptional contribution to the number of points.

We now proceed to find all the differential systems. As described above, we associate a period to each monomial as in (3.2), and the Picard–Fuchs equation for each representation can be found using repeatedly the operation

$$\begin{aligned} D_i\left(\frac{x^{\mathbf{v}}}{P^{w(\mathbf{v})+1}}\right) &\stackrel{\text{def}}{=} \partial_i\left(x_i\,\frac{x^{\mathbf{v}}}{P^{w(\mathbf{v})+1}}\right) \\ &= (1+v_i)\,\frac{x^{\mathbf{v}}}{P^{w(\mathbf{v})+1}} + 5(w(\mathbf{v})+1)\frac{x^{\mathbf{v}}}{P^{w(\mathbf{v})+2}}(x_i^5 - \psi Q)\ , \end{aligned} \tag{3.3}$$

where $x^{\mathbf{v}}$ is a monomial of degree $5w(\mathbf{v})$. Since

$$\int_\Gamma d^5x\, D_i\left(\frac{x^{\mathbf{v}}}{P^{w(\mathbf{v})+1}}\right) \;=\; 0$$

the derivative $D_i\left(\frac{x^{\mathbf{v}}}{P^{w(\mathbf{v})+1}}\right)$ corresponds to an exact form so (3.3) establishes identities between the differential form associated to the monomial $x^{\mathbf{v}}$ and those associated to $x^{\mathbf{v}}x_i^5$ and $x^{\mathbf{v}}Q$, up to an exact piece. Note in particular that the period corresponding to the monomial $x^{\mathbf{v}}Q$ corresponds to the derivative with respect to ψ of the period for the monomial $x^{\mathbf{v}}$. It will be very useful to represent the operations D_i by diagrams. For D_1, for example, the diagram is as follows and the other cases are similar

$$\begin{array}{ccc} (v_1, v_2, v_3, v_4, v_5) & \rightarrow & (v_1+1, v_2+1, v_3+1, v_4+1, v_5+1) \\ \downarrow D_1 & & \\ (v_1+5, v_2, v_3, v_4, v_5) & & \end{array}$$

.

Case 1

The monomials in this case are each invariant under $\mathcal{Z}$ and transform into each other under $\mathcal{P}$. The Picard–Fuchs equation for this system is well known[17] (since by itself it is the Picard–Fuchs equation for the *mirror* of the quintic Calabi–Yau manifold we are considering). Since we are writing for a mixed audience, and to set the stage for generalizations to Calabi–Yau manifolds in other dimensions, we rederive this differential equation here even though it is well known. We are interested in finding an identity, up to an exact form, between the differentials associated to the monomials Q^n. We obtain this identity using the operations D_i as above. Using the schematic representation of these operations, the desired relation can be found as shown below with the the monomials in the given representation indicated by bold face.

$$\begin{array}{ccccccccc} & & & & & & & & (4,4,4,4,-1) \rightarrow (5,5,5,5,0) \\ & & & & & & & & \downarrow D_5 \\ (\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) & \rightarrow & (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}) & \rightarrow & (\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2},\mathbf{2}) & \rightarrow & (\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3},\mathbf{3}) & \rightarrow & (\mathbf{4},\mathbf{4},\mathbf{4},\mathbf{4},\mathbf{4}) \\ \downarrow D_1 & & \downarrow D_1 & & \downarrow D_1 & & \downarrow D_1 & & \\ (5,0,0,0,0) & \rightarrow & (6,1,1,1,1) & \rightarrow & (7,2,2,2,2) & \rightarrow & (8,3,3,3,3) & & \\ \downarrow D_2 & & \downarrow D_2 & & \downarrow D_2 & & & & \\ (5,5,0,0,0) & \rightarrow & (6,6,1,1,1) & \rightarrow & (7,7,2,2,2) & & & & \\ \downarrow D_3 & & \downarrow D_3 & & & & & & \\ (5,5,5,0,0) & \rightarrow & (6,6,6,1,1) & & & & & & \\ \downarrow D_4 & & & & & & & & \\ (5,5,5,5,0) & & & & & & & & \end{array}$$

Note that we are allowing a (minimum) negative exponent of x_i of -1 when operating with D_i to allow for equivalences of monomials in the monomial ring. For example, the piece of the diagram above

$$\begin{array}{ccc} (4,4,4,4,-1) & \rightarrow & (5,5,5,5,0) \\ & \downarrow D_5 & \\ (4,4,4,4,4) & & \end{array}$$

means that the monomial Q^4 and $\psi\, x_1^5 x_2^5 x_3^5 x_4^5$ are equivalent in $\mathcal{I}$, the identity being

$$D_5\left(\frac{x_1^4 x_2^4 x_3^4 x_4^4}{x_5 P^4}\right) = \partial_5\left(\frac{x_1^4 x_2^4 x_3^4 x_4^4}{P^4}\right) = 5\frac{Q^4}{P^5} - 5\psi\frac{x_1^5 x_2^5 x_3^5 x_4^5}{P^5}\ .$$

Defining

$$Q_n = \frac{Q^{n-1}}{P^n}\ , \quad n \geq 1\ ,$$

we see that the identity we are looking for is of the form

$$\begin{aligned}
\psi Q_1 + A_2\psi^2 Q_2 + A_3\psi^3 Q_3 + A_4\psi^4 Q_4 + A_5\psi^5 Q_5 - \tilde{A}_5 Q_5 = \\
&\partial_1[x_1(B_{11}\psi Q_1 + B_{12}\psi^2 Q_2 + B_{13}\psi^3 Q_3 + B_{14}\psi^4 Q_4)] \\
&+ \partial_2[x_2 x_1^5 Q_1(B_{21}\psi Q_1 + B_{22}\psi^2 Q_2 + B_{23}\psi^3 Q_3)] \\
&+ \partial_3[x_3 x_1^5 x_2^5 Q_1^2(B_{31}\psi Q_1 + B_{32}\psi^2 Q_2)] \\
&+ \partial_4(x_4 x_1^5 x_2^5 x_3^5 B_{41}\psi Q_1^4) \\
&+ \partial_5[(x_1 x_2 x_3 x_4 B_5 Q_1)^4]\ ,
\end{aligned}$$

with the $A_j, \tilde{A}_5, B_{jk}$ and B_5 denoting certain constants which may be determined by evaluating the derivatives and comparing terms on each side of the equation. In this way we find

$$\begin{aligned}
\psi Q_1 + 75\psi^2 Q_2 + 1250\psi^3 Q_3 + 7500\psi^4 Q_4 + 15000(\psi^5 - 1)Q_5 = \\
&\psi\,\partial_1[x_1(Q_1 + 35\psi Q_2 + 300\psi^2 Q_3 + 750\psi^3 Q_4)] \\
&+ 5\psi\,\partial_2[x_2 x_1^5 Q_1(Q_1 + 30\psi Q_2 + 150\psi^2 Q_3)] \\
&+ 50\psi\,\partial_3[x_3 x_1^5 x_2^5 Q_1^2(Q_1 + 15\psi Q_2)] \\
&+ 750\psi\,\partial_4(x_4 x_1^5 x_2^5 x_3^5 Q_1^4) \\
&- 750\,\partial_5[(x_1 x_2 x_3 x_4 Q_1)^4]\ .
\end{aligned} \tag{3.4}$$

Consider now the period integrals

$$I_n = \frac{1}{(2\pi i)^5}\int_{\Gamma_0} d^5x\ Q_n\ ,$$

where Γ_0 is the lifting to $\mathbb{C}^5$ of the three cycle in M_ψ which is topologically a three torus in the large complex structure limit $\psi \to \infty$ and which intersects at a point with a three

sphere which is shrinking to a point near $\psi^5 = 1$ (a conifold point); Γ_0 is the five torus in $\mathbb{C}_5$ defined by $|x_i| = 1$. These integrals are all related to I_1 by

$$I_{n+1} = \frac{1}{5^n\, n!}\, \frac{\mathrm{d}^n}{\mathrm{d}\psi^n}\, I_1 \ .$$

The Picard–Fuchs equation for this case is the differential equation for I_1 that follows from (3.4)

$$\psi\, I_1 + 15\, \psi^2\, \frac{dI_1}{d\psi} + 25\, \psi^3\, \frac{d^2 I_1}{d\psi^2} + 10\, \psi^4\, \frac{d^3 I_1}{d\psi^3} + (\psi^5 - 1)\, \frac{d^4 I_1}{d\psi^4} = 0 \quad .$$

If we define $\varpi_0 = -5\, \psi\, I_1$, we obtain the usual form of the Picard–Fuchs equation

$$\mathcal{L}\, \varpi_0 \ = \ 0 \quad \text{with} \quad \mathcal{L} \ = \ \vartheta^4 - 5\lambda \prod_{i=1}^{4} (5\vartheta + i) \ , \tag{3.5}$$

where we have redefined the parameter to be $\lambda \ = \ (5\psi)^{-5}$, and have set $\vartheta = \lambda \frac{\mathrm{d}}{\mathrm{d}\lambda}$. Note that the form of the Picard–Fuchs equation depends on the gauge chosen for the holomorphic $(3,0)$–form, Ω. This gauge freedom corresponds to the fact that Ω is uniquely defined up to a holomorphic function of the complex structure of the parameters (that is, it defines a holomorphic line bundle over the moduli space). In the canonical gauge of equation (3.5), all four of its indices at $\lambda = 0$ are equal to zero, and the solutions are asymptotically like 1, $\log\lambda$, $\log^2\lambda$, and $\log^3\lambda$ as $\lambda \to 0$. The solution of (3.5) that is asymptotically 1, called the fundamental period $\varpi_0(\psi)$, is given by

$$\varpi_0(\psi) = {}_4F_3\left(\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}; 1, 1, 1; \psi^{-5}\right) = \sum_{m=0}^{\infty} a_m \lambda^m \ , \qquad a_m = \frac{(5m)!}{m!^5} \ .$$

A basis for the space of solutions of (3.5) can be found using the method of Frobenius. We review this procedure here to establish notation. Consider the series

$$F(\lambda, \varepsilon) = \sum_{m=0}^{\infty} A_m(\varepsilon) \lambda^{m+\varepsilon} \quad , \tag{3.6}$$

satisfying the equation $\mathcal{L}F(\lambda,\varepsilon) = \varepsilon^4 \lambda^\varepsilon$, and $F(\lambda, 0) = \varpi_0(\psi)$, so that $A_m(0) = a_m$. Then, the coefficients $A_m(\varepsilon)$ must satisfy $A_0(\varepsilon) = 1$ and the recurrence relations

$$A_m(\varepsilon) \ = \ \frac{(5m + 5\varepsilon)(5m + 5\varepsilon - 1) \ldots (5m + 5\varepsilon - 4)}{(m + \varepsilon)^5}\, A_{m-1}(\varepsilon) \ .$$

We choose

$$A_m(\varepsilon) \;=\; \frac{\Gamma(5m+5\varepsilon+1)}{\Gamma^5(m+\varepsilon+1)}\frac{\Gamma^5(\varepsilon+1)}{\Gamma(5\varepsilon+1)} \;. \tag{3.7}$$

The logarithmic solutions are then obtained by derivatives of $F(\lambda,\varepsilon)$ with respect to ε:

$$\varpi_k(\psi) = \frac{\partial^k}{\partial\varepsilon^k}F(\lambda,\varepsilon)\Big|_{\varepsilon=0} = \sum_{j=0}^{k}\binom{k}{j} f_j(\psi)\;\log^{k-j}\lambda\;, \quad k=0,\ldots,3\;,$$

where

$$f_j(\psi) = \sum_{m=0}^{\infty}\frac{\mathrm{d}^j}{\mathrm{d}\varepsilon^j}A_m(\varepsilon)\Big|_{\varepsilon=0}\lambda^m\;.$$

Cases 2-5: In each of these cases we have a set of two dimensional representations which are related to each other under $\mathcal{P}$, so that the periods corresponding to each one of these representations will satisfy the same differential equation. Given a representation the corresponding period integrals are

$$I_{\mathbf{v}} = \psi\left(\frac{1}{2\pi i}\right)^5\int_\Gamma d^5x\;\frac{x^{\mathbf{v}}}{P^2}\quad,\qquad I'_{\mathbf{v}} - I_{\mathbf{v}} = 10\psi^2\left(\frac{1}{2\pi i}\right)^5\int_\Gamma d^5x\;\frac{x^{\mathbf{v}}Q}{P^3}\;,$$

where we have chosen a convenient gauge. We will also need second derivatives of $I_{\mathbf{v}}$ so we also define

$$Q_{\mathbf{v},n} = \frac{x^{\mathbf{v}}Q^{n-1}}{P^{n+2}}\;,\quad n\geq 1\;.$$

Since all the cases at hand are similar, we now enumerate the results, which were obtained following a procedure analogous to that in **case 1**. For each case we give the diagrams which provide the identities for the differentials $Q_{\mathbf{v},n}d^5x$ associated to each monomial $x^{\mathbf{v}}$, the Picard–Fuchs equation for the periods and the solutions. As we will see, the Picard–Fuchs equations are, in all four cases, second order hypergeometric differential equations

$$\{\vartheta(\vartheta-1+c_{\mathbf{v}}) - 5^5\lambda(\vartheta+a_{\mathbf{v}})(\vartheta+b_{\mathbf{v}})\}I_{\mathbf{v}} = 0\;.$$

In all cases $c_{\mathbf{v}} = a_{\mathbf{v}} + b_{\mathbf{v}}$ which means that as $\psi^5\to 1$, there is a regular solution and a logarithmic solution. Also, in all cases there is a regular solution asymptotic to 1 as $\lambda\to 0$ given by

$$I_{\mathbf{v}} = {}_2F_1\left(a_{\mathbf{v}},b_{\mathbf{v}};c_{\mathbf{v}};\psi^{-5}\right) = \sum_{m=0}^{\infty}\lambda^m\frac{(5m)!}{\prod_{i=1}^5\left(\frac{5-v_i}{5}\right)_m}\;.$$

To find the second solution we need to distinguish between **cases 2** and **3** and **cases 4** and **5**. For **cases 2** and **3**, we have $c_{\mathbf{v}}=1$, and thus as $\lambda\to 0$ the second solution is

asymptotic to $\log\lambda$. We find this solution using the method of Frobenius, as in **case 1**, that is we consider the series

$$F(\lambda,\varepsilon) = \sum_{m=0}^{\infty} A_{\mathbf{v},m}(\varepsilon)\lambda^{m+\varepsilon} \quad ,$$

satisfying

$$\textbf{case 2} \ : \quad \left\{\vartheta^2 - 5^5\lambda\left(\vartheta+\tfrac{2}{5}\right)\left(\vartheta+\tfrac{3}{5}\right)\right\}F(\lambda,\varepsilon) = \varepsilon^2\lambda^{\varepsilon} \ , \quad \text{and} \quad F(\lambda,0) = I_{(4,1,0,0,0)}$$

$$\textbf{case 3} \ : \quad \left\{\vartheta^2 - 5^5\lambda\left(\vartheta+\tfrac{1}{5}\right)\left(\vartheta+\tfrac{4}{5}\right)\right\}F(\lambda,\varepsilon) = \varepsilon^2\lambda^{\varepsilon} \ , \quad \text{and} \quad F(\lambda,0) = I_{(3,2,0,0,0)}$$

so that

$$A_{\mathbf{v},m}(0) = \frac{(5m)!}{\prod_{i=1}^{5}(1-\frac{v_i}{5})_m} \ .$$

Then, the coefficients $A_{\mathbf{v},m}(\varepsilon)$ must satisfy $A_{\mathbf{v},0}(\varepsilon) = 1$ and the recurrence relations

$$A_{\mathbf{v},m}(\varepsilon) \ = \ 5^3\,\frac{(5m+5\varepsilon-5a_{\mathbf{v}})(5m+5\varepsilon-5b_{\mathbf{v}})}{(m+\varepsilon)^2}\,A_{\mathbf{v},m-1}(\varepsilon) \ .$$

Hence

$$A_{\mathbf{v},m}(\varepsilon) \ = \ \frac{\Gamma(5m+5\varepsilon+1)}{\Gamma(5\varepsilon+1)}\prod_{i=1}^{5}\frac{\Gamma\left(\varepsilon+1-\frac{v_i}{5}\right)}{\Gamma\left(m+\varepsilon+1-\frac{v_i}{5}\right)} \ .$$

The logarithmic solution is then obtained by taking the derivative of $F(\lambda,\varepsilon)$ with respect to ε:

$$J_{\mathbf{v}}(\psi) = \frac{\partial}{\partial\varepsilon}F(\lambda,\varepsilon)\Big|_{\varepsilon=0} = I_{\mathbf{v}}(\psi)\log\lambda + f_{1,\mathbf{v}}(\psi) \ ,$$

where

$$f_{1,\mathbf{v}}(\psi) = \sum_{m=0}^{\infty}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}A_{\mathbf{v},m}(\varepsilon)\Big|_{\varepsilon=0}\lambda^m \ .$$

For **cases 4** and **5**, $c_{\mathbf{v}}$ is *not* an integer, so the second solution, which tends asymptotically to $\psi^{5(-1+c_{\mathbf{v}})}$ as $\lambda\to 0$, is given by

$$\begin{aligned} J_{\mathbf{v}} \ &= \psi^{5(-1+c_{\mathbf{v}})}{}_2F_1\left(1-b_{\mathbf{v}},1-a_{\mathbf{v}};2-c_{\mathbf{v}};\psi^{-5}\right) \\ &= \psi^{5(-1+c_{\mathbf{v}})}\sum_{m=0}^{\infty}\lambda^m\frac{(5m)!}{(2-c_{\mathbf{v}})_m(b_{\mathbf{v}})_m(a_{\mathbf{v}})_m m!^2} \ . \end{aligned}$$

Case 2 : $\mathbf{v} = (4,1,0,0,0)$

Diagram:

$$\begin{array}{ccccc}
 & & & & (3,0,4,4,-1) \to (4,1,5,5,0) \\
 & & & & \downarrow D_5 \\
 & & & (2,-1,3,3,3) \to & (3,0,4,4,4) \\
 & & & \downarrow D_2 & \\
(-1,1,0,0,0) \to & (0,2,1,1,1) \to & (1,3,2,2,2) \to & (2,4,3,3,3) & \\
\downarrow D_1 & \downarrow D_1 & \downarrow D_1 & & \\
(\mathbf{4},\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0}) \to & (\mathbf{5},\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{1}) \to & (\mathbf{6},\mathbf{3},\mathbf{2},\mathbf{2},\mathbf{2}) & & \\
\downarrow D_3 & \downarrow D_3 & & & \\
(4,1,5,0,0) \to & (5,2,6,1,1) & & & \\
\downarrow D_4 & & & & \\
(4,1,5,5,0) & & & &
\end{array}$$

Identity for differential forms:

$$\begin{aligned}
&5\,(\psi^5-2)\,Q_{(4,1,0,0,0),1} + 50\,\psi\,(3\psi^5+2)\,Q_{(4,1,0,0,0),2} + 750\,\psi^2(\psi^5-1)\,Q_{(4,1,0,0,0),3} = \\
&\qquad 5\,\psi^5\partial_3(x_3(Q_{(4,1,0,0,0),1} + 10\,\psi Q_{(4,1,0,0,0),2})) + 50\,\psi^3\partial_4\left(x_4\ \frac{x_3^5}{P}Q_{(4,1,0,0,0),1}\right) \\
&\quad + 50\,\psi^4\partial_5\left(\frac{x_1^3x_3^4x_4^4}{P^3}\right) + 50\,\psi^3\partial_2\left(\frac{x_1^2x_3^3x_4^3x_5^3}{P^3}\right) \\
&\quad + 2\,\partial_1\left(\frac{x_2}{P} + x_1\left(-5\psi\,\frac{x_2^2x_3x_4x_5}{P^2} + 25\,\psi^2\,\frac{x_2^2x_3x_4x_5Q}{P^3}\right)\right)
\end{aligned}$$

Differential equation:

$$\left\{\vartheta^2 - 5^5\lambda\left(\vartheta+\tfrac{2}{5}\right)\left(\vartheta+\tfrac{3}{5}\right)\right\} I_{(4,1,0,0,0)} = 0$$

Holomorphic solution at $\lambda = 0$:

$$I_{(4,1,0,0,0)}(\psi) = {}_2F_1\left(\tfrac{2}{5},\tfrac{3}{5};1;\psi^{-5}\right) = \sum_{m=0}^{\infty}\lambda^m\frac{(5m)!}{(\frac{1}{5})_m(\frac{4}{5})_m m!^3}$$

Case 3 : $\mathbf{v} = (3,2,0,0,0)$

Diagram:

$$
\begin{array}{ccccc}
 & & & & (2,1,4,4,-1) \to (3,2,5,5,0) \\
 & & & & \downarrow D_5 \\
 & & (0,-1,2,2,2) & \to (1,0,3,3,3) & \to (2,1,4,4,4) \\
 & & \downarrow D_2 & & \downarrow D_2 \\
 & (-1,3,1,1,1) \to & (0,4,2,2,2) & \to (1,5,3,3,3) & \\
 & \downarrow D_1 & \downarrow D_1 & & \\
(\mathbf{3},\mathbf{2},\mathbf{0},\mathbf{0},\mathbf{0}) \to & (\mathbf{4},\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{1}) & \to (\mathbf{5},\mathbf{4},\mathbf{2},\mathbf{2},\mathbf{2}) & & \\
\downarrow D_3 & \downarrow D_3 & & & \\
(3,2,5,0,0) \to & (4,3,6,1,1) & & & \\
\downarrow D_4 & & & & \\
(3,2,5,5,0) & & & &
\end{array}
$$

Identity for differential forms:

$$
\begin{aligned}
&\psi^4\, Q_{(3,2,0,0,0),1} + 10\,(3\psi^5+2)\, Q_{(3,2,0,0,0),2} + 150\,\psi(\psi^5-1)\, Q_{(3,2,0,0,0),3} = \\
&\qquad \psi^4\, \partial_3(x_3(Q_{(3,2,0,0,0),1} + 10\,\psi Q_{(3,2,0,0,0),2})) + 10\,\psi^4 \partial_4 \left(x_4\ \frac{x_3^5}{P} Q_{(3,2,0,0,0),1} \right) \\
&+ 10\,\psi^3 \partial_5 \left(\frac{x_1^2 x_2 x_3^4 x_4^4}{P^3} \right) \\
&- 2\,\partial_1 \left(\frac{x_2^3 x_3 x_4 x_5}{P^2} - 5\psi x_1\, \frac{x_2^4 x_3^2 x_4^2 x_5^2}{P^3} \right) - \psi\, \partial_2 \left(\frac{x_3^2 x_4^2 x_5^2}{P^2} - 5\psi x_2\, \frac{x_1 x_3^3 x_4^3 x_5^3}{P^3} \right)
\end{aligned}
$$

Differential equation:

$$
\left\{ \vartheta^2 - 5^5 \lambda \left(\vartheta + \tfrac{1}{5}\right) \left(\vartheta + \tfrac{4}{5}\right) \right\} I_{(3,2,0,0,0)} = 0
$$

Holomorphic solution at $\lambda = 0$:

$$
I_{(3,2,0,0,0)}(\psi) = {}_2F_1\left(\tfrac{1}{5}, \tfrac{4}{5}; 1; \psi^{-5}\right) = \sum_{m=0}^{\infty} \lambda^m \frac{(5m)!}{(\frac{2}{5})_m (\frac{3}{5})_m m!^3}
$$

Case 4 : $\mathbf{v} = (3,1,1,0,0)$

Diagram:

$$\begin{array}{ccccc}
 & & & & (1,4,-1,3,3) \to (2,5,0,4,4) \\
 & & & & \downarrow D_3 \\
 & & & (0,3,3,2,2) & \to (1,4,4,3,3) \\
 & & & \downarrow D_1 & \\
 & & (\mathbf{3},\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0}) \to (\mathbf{4},\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{1}) & \to (\mathbf{5},\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{2}) & \\
 & & \downarrow D_4 \qquad\quad \downarrow D_4 & & \\
(2,0,0,4,-1) & \to & (3,1,1,5,0) \to (4,2,2,6,1) & & \\
\downarrow D_5 & & \downarrow D_5 & & \\
(2,0,0,4,4) & \to & (3,1,1,5,5) & & \\
\downarrow D_2 & & & & \\
(2,5,0,4,4) & & & &
\end{array}$$

Identity for differential forms:

$$\begin{aligned}
&2\,\psi^4 Q_{(3,1,1,0,0),1} + 10\,(4\psi^5+1) Q_{(3,1,1,0,0),2} + 150\,\psi(\psi^5-1)\,Q_{(3,1,1,0,0),3} \;= \\
&\quad 2\,\psi^4 \partial_4 (x_4 (Q_{(3,1,1,0,0),1} + 5\psi Q_{(3,1,1,0,0),2})) \\
&+ \psi^3 \partial_5 \left(\frac{x_1^2 x_4^4}{P^2} + 10\,\psi x_5 \frac{x_4^5}{P} Q_{(3,1,1,0,0),1} \right) + 10\,\psi^3 \partial_2 \left(x_2\, \frac{x_1^2 x_4^4 x_5^4}{P^3} \right) \\
&+ 10\,\psi^2\, \partial_3 \left(\frac{x_1 x_2^4 x_4^3 x_5^3}{P^3} \right) - \partial_1 \left(\frac{x_2^2 x_3^2 x_4 x_5}{P^2} - 10\,\psi x_1\, \frac{x_1 x_2^3 x_3^3 x_4^2 x_5^2}{P^3} \right)
\end{aligned}$$

Differential equation:

$$\left\{ \vartheta \left(\vartheta - \tfrac{1}{5}\right) - 5\lambda^5 \left(\vartheta + \tfrac{1}{5}\right) \left(\vartheta + \tfrac{3}{5}\right) \right\} I_{(3,1,1,0,0)} = 0$$

Holomorphic solution at $\lambda = 0$:

$$I_{(3,1,1,0,0)}(\psi) = {}_2F_1\left(\tfrac{1}{5}, \tfrac{3}{5}; \tfrac{4}{5}; \psi^{-5}\right) = \sum_{m=0}^{\infty} \lambda^m \frac{(5m)!}{(\frac{2}{5})_m (\frac{4}{5})_m^2 m!^2}$$

Case 5 : $\mathbf{v} = (2,2,1,0,0)$

Diagram:

$$\begin{array}{ccccccc}
 & & & & & & (-1,4,3,2,2) \to (0,5,4,3,3) \\
 & & & & & & \downarrow D_1 \\
 & & & & (\mathbf{2},\mathbf{2},\mathbf{1},\mathbf{0},\mathbf{0}) & \to & (\mathbf{3},\mathbf{3},\mathbf{2},\mathbf{1},\mathbf{1}) \to (\mathbf{4},\mathbf{4},\mathbf{3},\mathbf{2},\mathbf{2}) \\
 & & & & \downarrow D_4 & & \downarrow D_4 \\
 & & & (1,1,0,4,-1) & \to & (2,2,1,5,0) & \to (3,3,2,6,1) \\
 & & & \downarrow D_5 & & \downarrow D_5 & \\
 & (0,0,-1,3,3) & \to & (1,1,0,4,4) & \to & (2,2,1,5,5) & \\
 & \downarrow D_3 & & \downarrow D_3 & & & \\
 & (0,0,4,3,3) & \to & (1,1,5,4,4) & & & \\
 & \downarrow D_2 & & & & & \\
 & (0,5,4,3,3) & & & & &
\end{array}$$

Identity for differential forms:

$$\begin{aligned}
&3\,\psi^3 Q_{(2,2,1,0,0),1} + 50\,\psi^4 Q_{(2,2,1,0,0),2} + 150\,(\psi^5-1) Q_{(2,2,1,0,0),3} \;= \\
&\qquad \psi^3 \partial_4 (x_4 (3 Q_{(2,2,1,0,0),1} + 10\,\psi Q_{(2,2,1,0,0),2})) \\
&\quad + 2\,\psi^2 \partial_5 \left(\frac{x_1 x_2 x_4^4}{P^2} + 5\psi\, x_5\, \frac{x_4^5}{P} Q_{(2,2,1,0,0),1} \right) + \psi\, \partial_3 \left(\frac{x_4^3 x_5^3}{P^2} + 10\,\psi x_3\, \frac{x_1 x_2 x_4^4 x_5^4}{P^3} \right) \\
&\quad + 10\,\psi\, \partial_2 \left(x_2\, \frac{x_3^4 x_4^3 x_5^3}{P^3} \right) + 10\,\partial_1 \left(\frac{x_2^4 x_3^3 x_4^2 x_5^2}{P^3} \right)
\end{aligned}$$

Differential equation:

$$\left\{ \vartheta \left(\vartheta - \tfrac{2}{5} \right) - 5^5 \lambda \left(\vartheta + \tfrac{1}{5} \right) \left(\vartheta + \tfrac{2}{5} \right) \right\} I_{(2,2,1,0,0)} = 0$$

Holomorphic solution at $\lambda = 0$:

$$I_{(2,2,1,0,0)}(\psi) = {}_2F_1 \left(\tfrac{1}{5}, \tfrac{2}{5}; \tfrac{3}{5}; \psi^{-5} \right) = \sum_{m=0}^{\infty} \lambda^m \frac{(5m)!}{(\frac{3}{5})_m^2 (\frac{4}{5})_m m!^2}$$

3.2. The semiperiods

We shall first review briefly the properties of what might be termed the fundamental semiperiod integral pertaining to the quintic [17,18,19] then we will extend the notion of a semiperiod by associating one with each monomial in the variables $x_1, \ldots, x_5$.

As we have seen the fundamental period can be realised as a residue integral

$$\varpi_0 \;=\; \frac{5\psi}{(2\pi i)^4}\int_{\gamma_2\times\cdots\times\gamma_5}\frac{y_1 dy_2 dy_3 dy_4 dy_5}{P(y)}$$

with y_1 constant and the $\gamma_i,\ i = 2, \ldots, 5$, circles that enclose the origin. For ψ sufficiently large one of the integrals, say the y_5-integral, has a pole close to the origin and hence inside the contour. Consider the result of deforming the contours as in Figure 3.2. The rays are chosen so that $\sum_{i=1}^5 y_i^5$ is real and positive along them. We now use the representation

$$\frac{1}{P} \;=\; \int_0^\infty ds\, e^{-sP}$$

under the integral and then change the variables of integration from $\{s,\, y_2,\, y_3,\, y_4,\, y_5\}$ to $\{x_1,\, x_2,\, x_3,\, x_4,\, x_5\}$ by setting $x_i = s^{\frac{1}{5}} y_i,\ i = 1, \ldots, 5$, with y_1 treated as a constant. In this way we see that the fundamental period has the representation

$$\varpi_0 \;=\; \frac{5\psi}{(2\pi i)^4}\int_{\tilde\gamma_1\times\cdots\times\tilde\gamma_5} dx_1 dx_2 dx_3 dx_4 dx_5\, e^{-P(x,\psi)}$$

with the $\tilde\gamma_i$ as in the second of Figures 3.2. The integral converges for ψ sufficiently small.

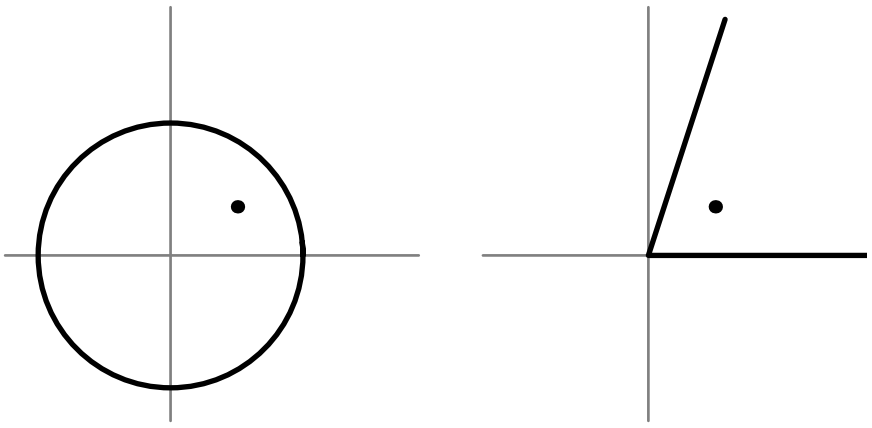

Figure 3.2: The loops γ_i may be deformed into pairs of rays running to infinity in the directions of fifth roots of unity.

It is therefore natural to consider integrals such as

$$\Upsilon_0(\psi) \;=\; 5\psi \int_0^\infty d^5x\, e^{-P(x,\psi)}$$

where the integration runs from 0 to ∞ for each of the variables x_i. We wish to show that $\Upsilon_0(\psi)$ is a semiperiod *i.e.* that it satisfies the differential equation $\mathcal{L}\Upsilon_0 = \text{constant}$. To this end take ψ to be small and expand the term $e^{-5\psi}\prod x$ in the integrand as a series:

$$\Upsilon_0(\psi) \;=\; 5\psi \sum_{n=0}^\infty \frac{(5\psi)^n}{n!}\left(\int_0^\infty d\xi\, e^{-\xi^5}\xi^n\right)^5 \;=\; \frac{1}{5^5}\sum_{n=0}^\infty \Gamma^5\left(\frac{n+1}{5}\right)\frac{\lambda^{-\left(\frac{n+1}{5}\right)}}{n!}\;. \qquad (3.8)$$

From this series it is immediate that

$$\mathcal{L}\Upsilon_0 \;=\; -\frac{1}{5^5}$$

and hence that $\vartheta\mathcal{L}\Upsilon_0 = 0$.

Now it is clear that all the functions $\Upsilon_0(\zeta^j\psi)$, with ζ a nontrivial fifth root of unity and $j = 0,\ldots,4$, satisfy the same equation. This gives a set of five semiperiods. The difference of any two satisfies the homogeneous Picard-Fuchs equation and hence is a period. We obtain in this way a full set of solutions to the Picard-Fuchs equation. The logarithms are not apparent in the series (3.8) since this series converges for small ψ. The logarithms appear when we analytically continue Υ_0 to large ψ and expand around $\psi = \infty$.

The function Υ_0 is a semiperiod associated with the fundamental period. We wish to discuss also semiperiods associate with the other periods. These other periods are associated with monomials $x^{\mathbf{v}}$ so let us consider integrals of the form

$$\Upsilon_{\mathbf{v}}(\psi) \;=\; 5\psi \int_0^\infty d^5x\, x^{\mathbf{v}} e^{-P(x,\psi)} \;=\; \frac{1}{5^5}\sum_{n=0}^\infty \frac{\prod_{i=1}^5 \Gamma\left(\frac{n+v_i+1}{5}\right)}{n!}\,\lambda^{-\left(\frac{n+1}{5}\right)}$$

These series converge for $|\psi| \le 1$. To obtain explicit expressions valid for large ψ one may continue the $\Upsilon_{\mathbf{v}}$ analytically. For our present purpose however it suffices to note that we can see from the series that the $\Upsilon_{\mathbf{v}}$ satisfy the differential equations $\mathcal{L}_{\mathbf{v}}\Upsilon_{\mathbf{v}} = 0$ with operators

$$\mathcal{L}_{\mathbf{v}} \;=\; \prod_{i=1}^5\left(\vartheta - \frac{v_i}{5}\right) - \lambda\prod_{i=1}^5(5\vartheta + i)\;. \qquad (3.9)$$

We can seek solutions to these equations that converge for large ψ *i.e.* small λ. These will contain logarithms so we initially seek series solutions of the form

$$\sum_{n=0}^\infty A_{\mathbf{v},n}(\varepsilon)\lambda^{n+\varepsilon}$$

with coefficients that can be taken to be

$$A_{\mathbf{v},n}(\varepsilon) \;=\; \frac{\Gamma(5n+5\varepsilon+1)}{\Gamma(5\varepsilon+1)} \prod_{i=1}^{5} \frac{\Gamma\left(\varepsilon+1-\frac{v_i}{5}\right)}{\Gamma\left(n+\varepsilon+1-\frac{v_i}{5}\right)} \;. \tag{3.10}$$

We saw previously that the periods associated with the monomials of cases 2 – 5 are in fact hypergeometric functions of second order. It is of interest to observe that the operator $\mathcal{L}_{\mathbf{v}}$ accommodates this by factorizing into the product of a third order operator and the second order operator corresponding to the hypergeometric equation. For the case $\mathbf{v} = (4,1,0,0,0)$ for example we have

$$\mathcal{L}_{(4,1,0,0,0)} \;=\; \vartheta\left(\vartheta-\tfrac{1}{5}\right)\left(\vartheta-\tfrac{4}{5}\right)\left\{\vartheta^2-5^5\lambda\left(\vartheta+\tfrac{2}{5}\right)\left(\vartheta+\tfrac{3}{5}\right)\right\}$$

and the operators corresponding to the other cases factor similarly.

4. $\nu(\psi)$ in Zeroth Order

4.1. Generalities

The first thing to note is that we work over $\mathbb{F}_p^5$ rather than the projective space $\mathbb{F}_p\mathbb{P}^4$ so the cardinality of the Calabi–Yau manifold is given by

$$N(\psi) \;=\; \frac{\nu(\psi)-1}{p-1}\ .$$

It is also often easier to work over $(\mathbb{F}_p^*)^5$, the set of points for which no x_i is zero. We therefore distinguish between

$$X_\psi \;=\; \left\{x\in\mathbb{F}_p^5 \mid P(x,\psi)=0\right\} \quad\text{and}\quad X_\psi^* \;=\; \left\{x\in(\mathbb{F}_p^*)^5 \mid P(x,\psi)=0\right\}$$

and denote by $\nu(\psi)$ and $\nu^*(\psi)$ the cardinality of these two sets.

Note that $(\mathbb{F}_p^*)^5$ is the disjoint union of the X_ψ^* since if $x\in(\mathbb{F}_p^*)^5$ then $\prod x_i\neq 0$ so

$$\frac{\sum_{i=1}^5 x_i^5}{\prod_{i=1}^5 x_i} \;=\; 5\,\psi$$

for some ψ and clearly no x can correspond to more than one ψ. Thus we have

$$\sum_{\psi\in\mathbb{F}_p}\nu^*(\psi) \;=\; (p-1)^5\ .$$

Consider now $x\in\mathbb{F}_p^5\setminus(\mathbb{F}_p^*)^5$, that is the set with at least one x_i zero. On this set the quintic reduces to $\sum_{i=1}^5 x_i^5=0$. Thus the set

$$Y \;=\; X_\psi\cap(\mathbb{F}_p^5\setminus(\mathbb{F}_p^*)^5)$$

is independent of ψ and

$$X_\psi \;=\; X_\psi^*\cup Y \;\text{ and }\; \nu(\psi) \;=\; \nu^*(\psi)+\nu_Y$$

with $\nu_Y=\mathrm{Card}(Y)$. It is easy to evaluate ν for the case that $5\not|\,(p-1)$ by calculating the number of points of Y with precisely j coordinates nonzero for $0\le j\le 4$. We find

$$\nu_Y \;=\; -4+10p-10p^2+5p^3 \;\;;\;\; 5\not|\,(p-1)\ .$$

It is also easy to calculate $\nu(0)$ for the case $5\not|\,(p-1)$ by simply observing that the equation $\sum_{i=1}^5 x_i^5=0$ can be solved by assigning arbitrary values to the variables x_1,x_2,x_3,x_4 and then solving uniquely for x_5 since there are no nontrivial fifth roots of unity when $5\not|\,(p-1)$. Hence

$$\nu(0) \;=\; p^4 \;\;;\;\; 5\not|\,(p-1)\ .$$

When $5|(p-1)$ expressions for ν_Y and $\nu(0)$ may be found in terms of Gauss sums or by expanding these quantities p-adically as in the following.

4.2. The calculation to zeroth order

We begin with the zeroth order calculation for the case that $5 \not| (p-1)$. This is elementary but a good introduction to the nature of the computation.

The number of points mod p is, as we see from (2.2), given by

$$\nu(\psi) \;=\; \sum_{x\in\mathbb{F}_p^5} \left(1 - P(x,\psi)^{p-1}\right) + \mathcal{O}(p) \;\; .$$

In this expression we expand P^{p-1} as binomial series and the powers of $\sum x_i^5$ using the multinomial series

$$\begin{aligned}\nu(\psi) &= p^5 - \sum_{x\in\mathbb{F}_p^5} \sum_{n=0}^{p-1} \sum_{\sum n_i = n} \frac{(p-1)!}{(p-1-n)!\prod_{i=1}^5 n_i!} (-5\psi)^{p-1-n} \prod_{i=1}^5 x_i^{5n_i-n+p-1} + \mathcal{O}(p) \\ &= -\sum_{x\in(\mathbb{F}_p^*)^5} \sum_{n=0}^{p-2} \sum_{\sum n_i = n} \frac{n!}{\prod_{i=1}^5 n_i!} (5\psi)^{-n} \prod_{i=1}^5 x_i^{5n_i-n} + \mathcal{O}(p)\end{aligned} \;\; .$$

Where, in passing to the second equality, we have recognised that

$$\frac{(p-1)!}{(p-1-n)!} \;=\; (p-1)(p-2)\ldots(p-n) \;=\; (-1)^n n! + \mathcal{O}(p)$$

as well as the fact that $a^{p-1} = 1 + \mathcal{O}(p)$ for $a \neq 0$. It is useful to recognise also that the $n = p-1$ term in the n sum corresponds to $\nu(0) = p^4$ and so may be omitted; also owing to the powers of $\prod_{i=1}^5 x_i$, in the remaining terms the x-sum can be restricted to $(\mathbb{F}_p^*)^5$. Now we use the fact that

$$\sum_{x_i\in\mathbb{F}_p} x_i^{5n_i-n} \;=\; \begin{cases} \mathcal{O}(p) & , \text{ if } 5n_i - n \neq 0 \\ -1 + \mathcal{O}(p) & , \text{ if } 5n_i - n = 0 \;. \end{cases} \tag{4.1}$$

So the $\mathcal{O}(1)$ term is obtained by setting

$$5n_i \;=\; n \;, \quad i = 1,\ldots,5 \;;$$

so the n_i are all equal, $n_i = m$ say, and $n = 5m$. Thus we find

$$\nu(\psi) \;=\; \sum_{m=0}^{\left[\frac{p}{5}\right]} \frac{(5m)!}{(m!)^5} \lambda^m + \mathcal{O}(p) \;=\; {}^{[p/5]}f_0(\lambda) + \mathcal{O}(p) \;\; .$$

The case $5|(p-1)$ is requires a more careful analysis but the result is the same to this order.

4.3. The zeroth order expression for a general CY hypersurface in a toric variety

With only a little more work an analogous result may be established for the case that M is a Calabi–Yau hypersurface in a toric variety. Such a hypersurface is described by a reflexive polyhedron Δ

$$P(x,\alpha) \;=\; \sum_{\mathbf{m}\in\Delta} \alpha_{\mathbf{m}}\, x^{\mathbf{m}} \;.$$

In general there will be $N+4$ coordinates x_i and P is homogeneous with respect to N scaling relations. Thus there are N degree vectors $\mathbf{h}_j$ such that the N multidegrees

$$\deg_j(\mathbf{m}) \;=\; \mathbf{h}_j\cdot\mathbf{m} \;=\; \mathbf{h}_j\cdot\mathbf{1}$$

are independent of $\mathbf{m}$. The polyhedron Δ is therefore a four-dimensional polyhedron embedded in an $N+4$ dimensional space. There are now many parameters $\alpha_{\mathbf{m}}$ but among them is a parameter $\alpha_{\mathbf{1}}$ analogous to ψ that multiplies the fundamental monomial $Q = x_1\dots x_{N+4}$. First we shall write an expression for the fundamental period $\varpi_0(\alpha)$ as a function of the coefficients $\alpha_{\mathbf{m}}$ and then we shall show that, mod p, the number of rational points, $\nu(\alpha)$, is given by a truncation of the fundamental period. We derive the relation $\varpi_0(\alpha) \equiv {}^{(p-1)}\varpi_0(\alpha)\varpi_0(\alpha^p)$ which is the generalization of (2.15).

First we write $\phi = -\alpha_{\mathbf{1}}$ and separate out the fundamental monomial

$$P(x,\alpha) \;=\; \widetilde{P}(x,\alpha) - \phi\, Q \qquad \text{with} \qquad \widetilde{P}(x,\alpha) \;=\; \sum_{\substack{\mathbf{m}\in\Delta\\ \mathbf{m}\neq\mathbf{1}}} \alpha_{\mathbf{m}} x^{\mathbf{m}}$$

then following [18] and our discussion of §3 we have

$$\varpi_0(\alpha) \;=\; \frac{\phi}{(2\pi i)^{N+4}}\oint \frac{d^{N+4}x}{\phi\, Q - \widetilde{P}} \;=\; \frac{1}{(2\pi i)^{N+4}}\oint \frac{d^{N+4}x}{Q}\left(1-\frac{\widetilde{P}}{\phi\, Q}\right)^{-1} \;.$$

In the integral on the right we expand the bracket which yields the expansion in inverse powers of ϕ:

$$\begin{aligned}\varpi_0(\alpha) \;&=\; \frac{1}{(2\pi i)^{N+4}}\sum_{n=0}^{\infty}\frac{1}{\phi^n}\oint\frac{d^{N+4}x}{Q}\frac{\widetilde{P}^n}{Q^n}\\ &=\; \frac{1}{(2\pi i)^{N+4}}\sum_{n=0}^{\infty}\frac{1}{\phi^n}\sum_{\sum_{\mathbf{m}} n_{\mathbf{m}}=n}\left(\prod_{\mathbf{m}}\alpha_{\mathbf{m}}^{n_{\mathbf{m}}}\right)\frac{n!}{\prod_{\mathbf{m}} n_{\mathbf{m}}!}\oint\frac{d^{N+4}x}{Q}\,x^{\sum_{\mathbf{m}} n_{\mathbf{m}}\mathbf{m}-n\mathbf{1}}\end{aligned}$$

where in this and the following equations the label $\mathbf{m}$ is understood to run over all monomials of Δ apart from $\mathbf{1}$. The effect of the contour integral is to pick out coefficients $n_{\mathbf{m}}$ such that

$$\sum_{\mathbf{m}} n_{\mathbf{m}}\,\mathbf{m} \;=\; n\mathbf{1}\ . \tag{4.2}$$

Note that this condition implies that $\sum_{\mathbf{m}} n_{\mathbf{m}} = n$, since the monomials all have the same multidegrees, so this condition need not be imposed separately. We are left with the following series for ϖ_0

$$\varpi_0(\alpha) \;=\; \sum_{n=0}^{\infty} \frac{n!}{\phi^n} \sum_{\substack{n_{\mathbf{m}} \\ \sum_{\mathbf{m}} n_{\mathbf{m}}\mathbf{m}=n\mathbf{1}}} \prod_{\mathbf{m}} \frac{\alpha_{\mathbf{m}}^{n_{\mathbf{m}}}}{n_{\mathbf{m}}!}\ . \tag{4.3}$$

There are now many sums in this series; however there is a principal sum over n and subordinate sums such that, if we truncate the principal sum, the subordinate sums are over a finite number of terms.

In order to compute $\nu(\alpha)$ mod p we proceed by summing $\left(1 - P^{p-1}\right)$ as in the previous subsection. After a few lines of calculation we find the expression

$$\nu(\alpha) \;=\; -\sum_{n=0}^{p-1} \frac{n!}{\phi^n} \sum_{\substack{n_{\mathbf{m}} \\ \sum_{\mathbf{m}} n_{\mathbf{m}}=n}} \prod_{\mathbf{m}} \frac{\alpha_{\mathbf{m}}^{n_{\mathbf{m}}}}{n_{\mathbf{m}}!} \sum_{x\in\mathbb{F}_p^{N+4}} x^{\sum_{\mathbf{m}} n_{\mathbf{m}}\mathbf{m}+(p-1-n)\mathbf{1}} + \mathcal{O}(p)\ .$$

The x-sum further restricts the $n_{\mathbf{m}}$ to those that satisfy

$$\sum_{\mathbf{m}} n_{\mathbf{m}}\mathbf{m} \;=\; n\mathbf{1} + (p-1)\mathbf{u}$$

with $\mathbf{u}$ an integral vector with nonnegative components. By taking degrees and comparing with the restriction $\sum_{\mathbf{m}} n_{\mathbf{m}} = n$ we see that $\deg(\mathbf{u}) = 0$ and hence that $\mathbf{u}$ vanishes. In this way we see that the $n_{\mathbf{m}}$ are restricted by (4.2). On taking into account the factor of $(p-1)^{N+4}$ that arises from the x-sum we see that

$$\nu(\alpha) \;=\; (-1)^{N-1\ (p-1)}\varpi_0(\alpha) + \mathcal{O}(p)\ .$$

This result holds irrespective of whether or not $p-1$ is divisible by the multidegrees of P.

4.4. A congruence involving the Frobenius map

Consider again the expression (4.3) for the fundamental period and break up the sums by writing

$$n_{\mathbf{m}} \;=\; r_{\mathbf{m}}\, p + s_{\mathbf{m}} \qquad \text{and} \qquad n \;=\; rp + s$$

with $0 \le s_{\mathbf{m}} \le p-1$ and $0 \le s \le p-1$. Now it will not, in general, be the case that

$$\sum r_{\mathbf{m}} = r \qquad \text{and} \qquad \sum s_{\mathbf{m}} = s \tag{4.4}$$

owing to the carries associated with the sum $\sum n_{\mathbf{m}}$. However the ratio $n!/\prod_{\mathbf{m}} n_{\mathbf{m}}!$ is $\mathcal{O}(1)$ precisely when there are no carries so to this order we may impose (4.4). Thus

$$\begin{aligned}
\varpi_0(\alpha) \;&=\; \sum_{\sum r_{\mathbf{m}}\,\mathbf{m}=r} \;\sum_{\sum s_{\mathbf{m}}\,\mathbf{m}=s} \frac{r!\,s!}{\phi^{rp+s}} \prod_{\mathbf{m}} \frac{\alpha_{\mathbf{m}}^{r_{\mathbf{m}}p}\,\alpha_{\mathbf{m}}^{s_{\mathbf{m}}}}{r_{\mathbf{m}}!\,s_{\mathbf{m}}!} + \mathcal{O}(p) \\
&=\; {}^{(p-1)}\varpi_0(\alpha)\,\varpi_0(\alpha^p) + \mathcal{O}(p) \;\;.
\end{aligned}$$

5. $\boldsymbol{\nu(\psi)}$ in First Order

5.1. The first order calculation

To calculate $\nu(\psi)$ mod p^2 we now use the formula

$$\nu(\psi) \;=\; \sum_{x\in\mathbb{P}^5_p} \Big(1 - P(x,\psi)^{p(p-1)}\Big) + \mathcal{O}\big(p^2\big) \quad ,$$

since if $P(x,\psi) \neq 0 \mod p$, then $P(x,\psi)^{(p-1)} = 1+\mathcal{O}(p)$, and $P(x,\psi)^{p(p-1)} = 1+\mathcal{O}\big(p^2\big)$.

A lengthy calculation which we do not give here yields $\nu(\psi)$ correct through $\mathcal{O}(p)$

$$\begin{aligned}\nu(\psi) \;=\;& {}^{[2p/5]}f_0(\lambda^p) + p\ {}^{[2p/5]}f_1'(\lambda^p) \\ & - \delta_p\, p \left\{ \sum_{m=0}^{k} \lambda^{mp} \sum_{\substack{\mathbf{v}\in\theta \\ 1\le \dim\theta \le 4}} \frac{(5m)!}{\prod_{i=1}^5 (m+v_i k)!} - \frac{20}{(2k)!(k!)^3} \right\} + \mathcal{O}\big(p^2\big)\end{aligned} \tag{5.1}$$

where $f_1'(\lambda^p)$ means $(\vartheta f_1)(\lambda^p)$ with $\vartheta = \lambda\frac{d}{d\lambda}$, and where δ_p distinguishes the cases that $5|(p-1)$

$$\delta_p \;=\; \begin{cases} 0\ , & \text{if } 5 \not| \ (p-1) \\ 1\ , & \text{if } 5 \mid (p-1) \end{cases}$$

and where one of the sums is taken over faces, θ, of Δ of dimension greater than one; that is the sum is taken over all quintic monomials apart from the interior point and the vertices of Δ.

There are perhaps two puzzles that present themselves with regard to the case that $5|p-1$. The first is that hypergeometric functions emerge that appear to be of fifth order rather than the second order hypergeometric functions that we might have expected from our discussion of the periods. The second puzzle is that the sum over the points of Δ includes points interior to codimension one faces which do not contribute to the count of the periods. We shall see that both of these difficulties disappear on closer inspection.

5.2. The sum over monomials

For the case $5|p-1$ let us consider the sum over monomials $\mathbf{v} \in \Delta$. It is sufficient to list these up to permutation of the components which we do in the table below. In the table it is the monomial $(2,1,1,1,0)$ together with its permutations that lies in the interior of the codimension one faces.

monomial	permutations	$\{a_{\mathbf{v}},\, b_{\mathbf{v}};\, c_{\mathbf{v}}\}$
(4,1,0,0,0)	20	$\{\frac{2}{5},\, \frac{3}{5};\, 1\}$
(3,2,0,0,0)	20	$\{\frac{1}{5},\, \frac{4}{5};\, 1\}$
(3,1,1,0,0)	30	$\{\frac{1}{5},\, \frac{3}{5};\, \frac{4}{5}\}$
(2,2,1,0,0)	30	$\{\frac{1}{5},\, \frac{2}{5};\, \frac{3}{5}\}$
(2,1,1,1,0)	20	

Table 5.1

Consider now the new terms that appear when $5|p-1$. These can be written as above which emphasizes the simple dependence of these terms on the monomials of Δ. On the other hand we may rewrite these terms to show that they correspond to the periods that we met in §3. In order to simplify the expression further we note that

$$(m+kv_i)! \;=\; (1+kv_i)_m\;(kv_i)! \;=\; \Big(1-\frac{v_i}{5}\Big)_m\;(kv_i)! + \mathcal{O}(p) \tag{5.2}$$

where $(a)_m = \Gamma(a+m)/\Gamma(a) = a(a+1)\ldots(a+m-1)$ denotes the Pochhammer symbol. Before proceeding note that this shows that to the required order the term corresponds to a fifth order differential equation with operator $\mathcal{L}_{\mathbf{v}}$ as in (3.9).

By using (5.2) and also the multiplication formula (2.13) we may rewrite the terms enclosed in braces in (5.1) in the form

$$\sum_{\mathbf{v}} \frac{\gamma_{\mathbf{v}}}{\prod_{i=1}^5 (v_i k)!} \sum_{m=0}^{k} \frac{\prod_{i=1}^5 \big(\frac{i}{5}\big)_m}{\prod_{i=1}^5 \big(1-\frac{v_i}{5}\big)_m}\, \psi^{-5m} - \frac{20}{(2k)!(k!)^3} + \mathcal{O}(p) \tag{5.3}$$

where the coefficients $\gamma_{\mathbf{v}}$ account for the permutation factors of the table. For a given $\mathbf{v}$ there is cancellation between the Pochhammer symbols in the numerator and denominator.

Consider the contribution of the monomial $(4,1,0,0,0)$. The ratio of Pochhammer symbols reduces to $(2/5)_m\,(3/5)_m\,/(m!)^2$. So the contribution of this monomial reduces to the truncation of a second order hypergeometric function

$$\frac{20}{(4k)!k!}\sum_{m=0}^{k}\frac{\left(\frac{2}{5}\right)_m\left(\frac{3}{5}\right)_m}{(m!)^2}\,\psi^{-5m} \;=\; \frac{20}{(4k)!k!}\,{}_2^kF_1\big(\tfrac{2}{5},\tfrac{3}{5};1;\,\psi^{-5}\big)\;.$$

In this way the contribution of each $\mathbf{v}$, apart from $(2,1,1,1,0)$, reduces to the form

$$\frac{\gamma_{\mathbf{v}}}{\prod_{i=1}^{5}(v_ik)!}\,{}_2^kF_1\big(a_{\mathbf{v}},b_{\mathbf{v}};c_{\mathbf{v}};\,\psi^{-5}\big)$$

and we list the parameters of the hypergeometric functions in Table 5.1.

We turn now to discuss the contribution of the discrepant monomial $(2,1,1,1,0)$ which appears to be related to a ${}_3F_2$ but turns out to be a prolongation of the contribution of $(4,1,0,0,0)$. As a preliminary we note some elementary congruences. First by writing

$$(p-1)! \;=\; (p-1)(p-2)\dots(p-k)\times(4k)! \;=\; (p-1)(p-2)\dots(p-2k)\times(3k)!$$

we see that

$$(4k)!\,k! \;=\; -1+\mathcal{O}(p)\quad\text{and}\quad (3k)!\,(2k)! \;=\; -1+\mathcal{O}(p)\;.$$

We may also write

$$\Gamma(a+\mu) \;=\; \frac{\Gamma(a+\mu)}{\Gamma(a)}\,\Gamma(a) \;=\; (a)_\mu\,\Gamma(a)$$

and applying this to $(ak)!$ we find

$$(ak)! \;=\; ((a-1)k+1)_k\,\big((a-1)k\big)! \;=\; \left(1-\frac{(a-1)}{5}\right)_k\big((a-1)k\big)!+\mathcal{O}(p)\;\;;\;\;2\le a\le 5$$

from which we see that

$$(ak)! \;=\; k!\prod_{i=1}^{a-1}\left(1-\frac{i}{5}\right)_k+\mathcal{O}(p)\;.$$

Turn now to the contribution of the monomial $(2,1,1,1,0)$. First note that the extra constant in (5.3) is just such as to remove the $m=0$ term. Thus we are concerned with the sum

$$\frac{20}{(2k)!\,(k!)^3}\sum_{m=1}^{k}\frac{\left(\frac{1}{5}\right)_m\left(\frac{2}{5}\right)_m}{\left(\frac{4}{5}\right)_m^2}\,\psi^{-5m}\;. \tag{5.4}$$

Consider now the prolongation of the contribution of the monomial $(4,1,0,0,0)$ to $2k$ terms.

$$\begin{aligned}\frac{20}{(4k)!\,k!}\sum_{m=1}^{k}\frac{\left(\frac{2}{5}\right)_{m+k}\left(\frac{3}{5}\right)_{m+k}}{(1)_{m+k}^{2}}\,\psi^{-5m} &= -\frac{\left(\frac{2}{5}\right)_{k}\left(\frac{3}{5}\right)_{k}}{(k!)^{2}}\sum_{m=1}^{k}\frac{\left(k+\frac{2}{5}\right)_{m}\left(k+\frac{3}{5}\right)_{m}}{(k+1)_{m}^{2}}\,\psi^{-5m}\\ &= \frac{20}{(2k)!\,(k!)^{3}}\sum_{m=1}^{k}\frac{\left(\frac{1}{5}\right)_{m}\left(\frac{2}{5}\right)_{m}}{\left(\frac{4}{5}\right)_{m}^{2}}\,\psi^{-5m}+\mathcal{O}(p)\end{aligned}$$

which is the same as (5.4) to the required order and where in passing to the last equality we have used the congruences above. To write the contributions uniformly we may now extend the summation as far as $m=p-1$ since the additional terms are $\mathcal{O}\big(p^2\big)$

$$\begin{aligned}\nu(\psi) = {}& {}^{(p-1)}f_0(\lambda^p)+p\ {}^{(p-1)}f_1'(\lambda^p)\\ &-\delta_p\,p\sum_{\mathbf{v}}\frac{\gamma_{\mathbf{v}}}{\prod_{i=1}^{5}(v_ik)!}\ {}^{(p-1)}{}_2F_1\big(a_{\mathbf{v}},b_{\mathbf{v}};c_{\mathbf{v}};\,\psi^{-5}\big)+\mathcal{O}\big(p^2\big)\end{aligned}\tag{5.5}$$

where the monomial $(2,1,1,1,0)$ is now omitted from the $\mathbf{v}$-sum.

6. $\nu(\psi)$ in Higher Order and for Finer Fields

In this section we will make an ansatz for $\nu(\psi)$ for the case that $5 \not| \, p-1$. Which we are then able to compare with the numbers obtained by counting solutions to the equation $P = 0$. It turns out that this process is considerably simpler for the case $5 \not| \, p-1$ since then we do not have to take into account the contributions of the other periods. It will be natural then to extend the ansatz to the case of $\mathbb{F}_q$ with $q = p^s$ providing $5 \not| \, q-1$. Note that 5 will divide $p^s - 1$ for some values of s even if $5 \not| \, p-1$. In fact it is easy to see that $5|p^4 - 1$ for all $p \neq 5$ so this restriction is quite serious. Fortunately we will be able to compute $\nu(\psi)$ in terms of Gauss sums in §9 for the case that $5|q-1$ as well as the case $5 \not| \, q-1$. This will in fact establish the expressions that we propose here. We prefer however to proceed in the following way since this emphasizes the role of the periods and this is somewhat harder to see when one works with the Gauss sums.

6.1. An ansatz for the case $5 \not| \, p-1$

For the case $5 \not| \, (p-1)$ we can guess the form of $\nu(\psi)$ to next order

$$\nu(\psi) \;=\; (1 + A\,p^2)^{\,[3p/5]} f_0(\lambda^{p^2}) + p(1 + B\,p)^{\,[3p/5]} f_1'(\lambda^{p^2}) + C\,p^2\,{}^{[3p/5]} f_2''(\lambda^{p^2}) + \mathcal{O}(p^3)$$

and fix the constants by comparing with the value of $\nu(\psi)$ found by counting points with a computer for various values of p. Having fixed these constants we make a similar ansatz at next order and so on. In this way we find what appears to be the exact expression for $\nu(\psi)$

$$\begin{aligned}\nu(\psi) \;=\; & {}^{(p-1)} f_0(\lambda^{p^4}) + \left(\frac{p}{1-p}\right)\,{}^{(p-1)} f_1'(\lambda^{p^4}) + \frac{1}{2!}\left(\frac{p}{1-p}\right)^2\,{}^{(p-1)} f_2''(\lambda^{p^4}) \\ & + \frac{1}{3!}\left(\frac{p}{1-p}\right)^3\,{}^{(p-1)} f_3'''(\lambda^{p^4}) + \frac{1}{4!}\left(\frac{p}{1-p}\right)^4\,{}^{(p-1)} f_4''''(\lambda^{p^4}) \qquad (\mathrm{mod}\, p^5)\end{aligned}$$

The most interesting aspect is the appearance of the semiperiod. There is a procedure for finding the differential equations satisfied by the periods directly from the reflexive polyhedron ∇ corresponding to the manifold; this yields the GKZ system. This does not quite give the PF equation. For our case the PF operator is

$$\mathcal{L} \;=\; \vartheta^4 - 5\lambda \prod_{i=1}^{4} (5\vartheta + i) \quad , \qquad \text{with} \quad \vartheta = \lambda \frac{d}{d\lambda}$$

while the GKZ operator is

$$\mathcal{L}^\nabla \;=\; \vartheta\,\mathcal{L} \;=\; \vartheta^5 - \lambda\prod_{i=1}^{5}(5\vartheta+i)\ .$$

All the solutions to the PF equation satisfy the higher order equation, but there is now an extra solution

$$\varpi_4(\lambda) \;=\; f_0(\lambda)\,\log^4\lambda + 4\,f_1(\lambda)\,\log^3\lambda + 6\,f_2(\lambda)\,\log^2\lambda + 4\,f_3(\lambda)\,\log\lambda + f_4(\lambda)$$

The semiperiod is also an integral of Ω

$$\varpi_4(\lambda) \;=\; \int_C \Omega\ , \quad \text{but} \quad \partial C \neq 0\ .$$

6.2. The Method of Frobenius

We have to be clear now about the basis for the periods. Saying that they have the form

$$\begin{aligned}\varpi_0 \;&=\; f_0\\ \varpi_1 \;&=\; f_0\,\log\lambda + f_1 \quad \text{etc.}\end{aligned}$$

does not specify the f_k uniquely. We can pick a particular basis using the method of Frobenius. We seek a series solution to the GKZ equation $\mathcal{L}^\nabla F(\lambda,\varepsilon) = \varepsilon^5$ of the form

$$F(\lambda,\varepsilon) \;=\; \sum_{m=0}^{\infty} A_m(\varepsilon)\,\lambda^{m+\varepsilon}$$

and our periods are obtained as the derivatives of $F(\lambda,\varepsilon)$ with respect to ε. We work mod ε^5 and expand

$$F(\lambda,\varepsilon) \;=\; \sum_{k=0}^{4}\frac{\varepsilon^k}{k!}\,\varpi_k(\lambda)\ .$$

The coefficients satisfy the recurrence

$$A_m(\varepsilon) \;=\; \frac{(5m+5\varepsilon)(5m+5\varepsilon-1)\ldots(5m+5\varepsilon-4)}{(m+\varepsilon)^5}\,A_{m-1}(\varepsilon)$$

so we choose

$$A_m(\varepsilon) \;=\; \frac{\Gamma(5m+5\varepsilon+1)}{\Gamma^5(m+\varepsilon+1)}\frac{\Gamma^5(\varepsilon+1)}{\Gamma(5\varepsilon+1)} \qquad (6.1)\ .$$

Differentiating $A_m(\varepsilon)$ with respect to ε we have

$$A'_m(\varepsilon) \;=\; 5\Phi_m(\varepsilon)A_m(\varepsilon) \;\;,\quad A''_m(\varepsilon) \;=\; 5\left(\Phi'_m(\varepsilon)+5\Phi^2(\varepsilon)\right)A_m(\varepsilon) \;\;,\quad \text{etc.}$$

where Φ_m can be written in terms of the digamma function

$$\begin{aligned}\Phi_m(\varepsilon) \;&=\; \big(\Psi(5m+5\varepsilon+1)-\Psi(5\varepsilon+1)\big)-\big(\Psi(m+\varepsilon+1)-\Psi(\varepsilon+1)\big)\\ &=\; \sum_{j=1}^{5m}\frac{1}{j+5\varepsilon}-\sum_{j=1}^{m}\frac{1}{j+\varepsilon}\end{aligned}$$

Thus the $A_m^{(\ell)}(0)$ are of the form $b_{m\ell}a_m$ with $b_{m\ell}$ a polynomial in the

$$\Phi_m^{(k)}(0) \;=\; (-1)^k k!\left(5^k\sigma_{5m}^{(k+1)}-\sigma_m^{(k+1)}\right)$$

with $0\le k\le \ell$ and with the $\sigma_m^{(\ell)}$ as in (2.5).

We work mod ε^5 and expand

$$A_m(\varepsilon) \;=\; \sum_{k=0}^{4}\frac{1}{k!}\,\varepsilon^k a_m b_{mk} \tag{6.2}$$

then the f_k are simply

$$f_k \;=\; \sum_{m=0}^{\infty} a_m b_{mk}\lambda^m \;. \tag{6.3}$$

The extra Γ-functions that appear in the A_m serve to remove transcendental quantities $\Psi'(1),\ \Psi''(1)$ etc. that would otherwise appear in the coefficients b_{mk}.

Before proceeding note a useful identity. In the definition (6.1) set $m=s$ and $\varepsilon=rp$. We see that

$$a_{rp+s} \;=\; a_{rp}\,A_s(rp)$$

which we may regard as the exact form of the congruence (2.14). (Note that we work mod p^5 in order to calculate $\nu(\psi)$.)

Returning to our discussion of the choice of basis note that we may replace the A_m by another choice. We can parametrize this freedom by means of a function $h(\varepsilon)$ with $h(0)=1$

$$\widetilde{A}_m(\varepsilon) \;=\; h(\varepsilon)\,A_m(\varepsilon) \;\;\text{with}\;\; h(\varepsilon) \;=\; \sum_{k=0}^{4}\frac{1}{k!}\,h_k\,\varepsilon^k \;.$$

This yields another basis:

$$\begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \\ \tilde{f}_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ h_1 & 1 & 0 & 0 & 0 \\ h_2 & 2h_1 & 1 & 0 & 0 \\ h_3 & 3h_2 & 3h_1 & 1 & 0 \\ h_4 & 4h_3 & 6h_2 & 4h_1 & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}$$

In the f_k basis our result for $\nu(\psi)$ takes the form

$$\begin{aligned} \nu(\psi) \;=\; & {}^{(p-1)}f_0(\lambda^{p^4}) + \ldots + \frac{1}{4!}\left(\frac{p}{1-p}\right)^4 {}^{(p-1)}f_4^{(4)}(\lambda^{p^4}) \\ & + C_3\left(\frac{p}{1-p}\right)^3 \left\{ {}^{(p-1)}f_0^{(3)}(\lambda^{p^4}) + \left(\frac{p}{1-p}\right) {}^{(p-1)}f_1^{(4)}(\lambda^{p^4}) \right\} + \mathcal{O}\big(p^5\big) \end{aligned} \tag{6.4}$$

where C_3 is a constant (depending on p) that, on the basis of numerical experimentation, we identify as

$$C_3 \;=\; 240\,\frac{\sigma_{p-1}}{p^2} + \mathcal{O}\big(p^2\big) \;\; .$$

The 'ugly' terms in (6.4) may be removed by a change of basis corresponding to the function

$$h(\varepsilon) \;=\; 1 + \frac{C_3}{3!}\,\varepsilon^3 + \mathcal{O}\big(\varepsilon^5\big) \;\; .$$

This h is interesting because it is the same function that appears in the expansion

$$\frac{a_{rp}}{a_r} \;=\; h(rp) \;\; . \tag{6.5}$$

We can therefore rewrite our expression for the number of points

$$\begin{aligned} \nu(\psi) \;&=\; \sum_{s=0}^{p-1}\sum_{k=0}^{4} \frac{1}{k!}\left(\frac{ps}{1-p}\right)^k b_{sk}\, a_s\, h\left(\frac{ps}{1-p}\right) \lambda^{sp^4} \\ &=\; \sum_{s=0}^{p-1} A_s\left(\frac{ps}{1-p}\right) \frac{a_{sp(1+p+p^2+p^3)}}{a_{s(1+p+p^2+p^3)}}\,\lambda^{sp^4} \\ &=\; \sum_{s=0}^{p-1} \frac{a_{s(1+p+p^2+p^3+p^4)}}{a_{s(1+p+p^2+p^3)}}\,\lambda^{sp^4} \end{aligned} \tag{6.6}$$

the right hand side being understood as evaluated mod p^5.

We have succeeded in finding an expression for the exact number by evaluating the expressions in (6.4) and (6.6) mod p^5. We can however do better by seeking an exact p-adic expression. In virtue of our discussion of §2.5 we recognise λ^{p^4} as the appropriate approximation to Teich(λ). It is therefore natural to suppose that this is given by

$$\nu(\psi) \;=\; \sum_{m=0}^{p-1} \beta_m \,\mathrm{Teich}^m(\lambda) \tag{6.7}$$

with

$$\beta_m \;=\; \lim_{n\to\infty} \frac{a_{m(1+p+p^2+\ldots+p^{n+1})}}{a_{m(1+p+p^2+\ldots+p^n)}} \;. \tag{6.8}$$

We shall shortly evaluate this limit. We can however also retrace the steps that led to (6.7) to write expressions for $\nu(\psi)$ in terms of an extended set of semiperiods that will obtain mod p^6, mod p^7 and so on. To do this we simply forget the restriction $\varepsilon^5 = 0$ and expand $A_m(\varepsilon)$ as in (6.1) but now to all orders in ε

$$A_m(\varepsilon) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\,\varepsilon^k a_m b_{mk} \tag{6.9}$$

and through (6.3) this defines semiperiods f_j for all j. These semiperiods satisfy the equations

$$\vartheta^{j-3}\mathcal{L}\, f_j \;=\; 0 \quad , \quad \text{for } j \geq 3.$$

Returning to the evaluation of the coefficients β_m it is useful also to write the ratio a_{rp}/a_r as in (6.5) in terms of the p-adic Γ-function. From the relation (2.8) it is easy to see that, for $r \in \mathbb{Z}$, $r \geq 0$,

$$\frac{a_{rp}}{a_r} \;=\; \frac{\Gamma_p(5rp+1)}{\Gamma_p^5(rp+1)}$$

and the right hand member is analytic in the p-adic sense and so tends to the limit

$$\frac{\Gamma_p\left(\frac{5mp}{1-p}+1\right)}{\Gamma_p^5\left(\frac{mp}{1-p}+1\right)} \tag{6.10}$$

as $r \to \frac{m}{1-p}$. Dwork, writing under the pseudonym of Boyarski [20], has given the following useful series for $\Gamma_p(1+pZ)$, $Z \in \mathbb{Z}_p$

$$\Gamma_p(1+pZ) \;=\; -\Gamma_p(pZ) \;=\; -\sum_{n=0}^{\infty} c_{np}\, p^n\, (Z)_n \tag{6.11}$$

where $(Z)_n$ is again the Pochhammer symbol, and the coefficients are defined via the function

$$F(X) \;=\; \exp\left(X + \frac{X^p}{p}\right) \;=\; \sum_{n=0}^{\infty} c_n X^n \;. \tag{6.12}$$

By differentiating $F(X)$ it is easy to see that the c_n satisfy the recurrence

$$nc_n \;=\; c_{n-1} + c_{n-p} \quad , \quad c_0 \;=\; 1 \quad , \quad c_n \;=\; 0 \;\text{ for }\; n < 0 \;. \tag{6.13}$$

These relations permit rapid numerical evaluation of the limit (6.10). We will return in §9 to discuss the remarkable properties of this function.

Now the remaining factor that we need to consider to calculate β_m is

$$A_m\left(\frac{mp}{1-p}\right) \;=\; \frac{\Gamma\left(5m + \frac{5mp}{1-p} + 1\right)\Gamma\left(\frac{mp}{1-p}+1\right)^5}{\Gamma\left(\frac{5mp}{1-p}+1\right)\Gamma\left(m + \frac{mp}{1-p} + 1\right)^5} \;=\; \frac{\prod_{j=1}^{5m}\left(\frac{5mp}{1-p} + j\right)}{\prod_{j=1}^{m}\left(\frac{mp}{1-p}+j\right)^5} \tag{6.14}$$

and the right member is seen to be a rational number. Thus

$$\beta_m \;=\; \frac{\Gamma_p\left(\frac{5mp}{1-p}+1\right)}{\Gamma_p^5\left(\frac{mp}{1-p}+1\right)} \frac{\prod_{j=1}^{5m}\left(\frac{5mp}{1-p}+j\right)}{\prod_{j=1}^{m}\left(\frac{mp}{1-p}+j\right)^5} \;. \tag{6.15}$$

The series (6.7) for $\nu(\psi)$ has two constant terms corresponding to $m = 0$ and $m = p-1$. For these values of m we have

$$\beta_0 \;=\; 1 \quad , \quad \beta_{p-1} \;=\; p^4 \tag{6.16}$$

while for $1 \le m \le p-2$ we may write

$$\beta_m \;=\; (-p)^{\left[\frac{5m}{p-1}\right]} \frac{\Gamma_p\left(1 - \langle\frac{5m}{p-1}\rangle\right)}{\Gamma_p^5\left(1 - \langle\frac{m}{p-1}\rangle\right)} \tag{6.17}$$

where $\langle..\rangle$ denotes the fractional part. This last expression may be derived from (2.7). To see this take $p > 5$ and $0 < t < 5(p-1)$ an integer such that $p - 1 \nmid t$. Set therefore $t = r(p-1) + \mu$ with $1 \le \mu \le p-2$. Then, by separating the terms in the product

$$\prod_{j=1}^{t}\left(j + \frac{pt}{1-p}\right)$$

into those that are divisible by p and those that are not, we see that

$$\Gamma_p\left(1+\frac{pt}{1-p}\right)\prod_{j=1}^{t}\left(j+\frac{pt}{1-p}\right)=(-1)^t\,p^{\left[\frac{t}{p}\right]}\,\Gamma_p\left(1+\frac{t}{1-p}\right)\prod_{j=1}^{\left[\frac{t}{p}\right]}\left(j+\frac{t}{1-p}\right)$$

$$=(-1)^{\mu+r}\,p^{\left[\frac{t}{p}\right]}\,\Gamma_p\left(1+\frac{\mu}{1-p}\right)\frac{\prod_{j=1}^{\left[\frac{t}{p}\right]}\left(j-r+\frac{\mu}{1-p}\right)}{\prod_{j=1}^{r}{}'\left(j-r+\frac{\mu}{1-p}\right)}\ .$$

Now

$$\left[\frac{t}{p}\right]\ =\ \left[r+\frac{\mu-r}{p}\right]\ =\ \begin{cases} r & \text{if } \mu\geq r,\\ r-1 & \text{if } \mu<r,\end{cases}$$

and we also have

$$\frac{\prod_{j=1}^{\left[\frac{t}{p}\right]}\left(j-r+\frac{\mu}{1-p}\right)}{\prod_{j=1}^{r}{}'\left(j-r+\frac{\mu}{1-p}\right)}\ =\ \begin{cases}1 & \text{if } \mu\geq r,\\ p & \text{if } \mu<r,\end{cases}$$

since, when $\mu<r$, the omission of terms divisible by p causes the term with $j=r-\mu$ to be omitted from the denominator. In this way we see that

$$\Gamma_p\left(1+\frac{pt}{1-p}\right)\prod_{j=1}^{t}\left(j+\frac{pt}{1-p}\right)\ =\ (-1)^t\,(-p)^{\left[\frac{t}{p-1}\right]}\,\Gamma_p\left(1-\left\langle\frac{t}{p-1}\right\rangle\right)\tag{6.18}$$

and from this (6.17) follows immediately. As a matter of practical computation it is useful to note that Eq. (6.17) shows that it is necessary only to evaluate $\Gamma_p\left(1-\frac{t}{p-1}\right)$ for $0\leq t\leq p-2$ and for such t's Eq. (6.18), read from right to left, together with Boyarski's series (6.11) provide an efficient means of computation.

We note from (6.17) the interesting relation

$$\beta_{p-1-m}\ =\ (-p)^{4-\left[\frac{5m}{p-1}\right]}\,\frac{\Gamma_p\left(\left\langle\frac{5m}{p-1}\right\rangle\right)}{\Gamma_p^5\left(\left\langle\frac{m}{p-1}\right\rangle\right)}\ =\ \frac{p^4}{\beta_m}\ .$$

The first equality is true for $0\leq m\leq p-2$ but owing to (6.16) the relation $\beta_{p-1-m}=p^4/\beta_m$ is true also for $m=p-1$. In passing to the second equality in the equation above we have used the reflection formula (2.9).

It is easy to see also that $\beta_{\frac{p-1}{2}}=p^2$ for $p\geq 3$ so we can write $\nu(\psi)$ in the form

$$\begin{aligned}\nu(\psi)&=1+p^4+\sum_{m=1}^{p-2}\beta_m\mathrm{Teich}^m(\lambda)\\ &=1+p^4+p^2\mathrm{Teich}^{\frac{p-1}{2}}(\lambda)+\sum_{m=1}^{\frac{p-3}{2}}\left(\beta_m\mathrm{Teich}^m(\lambda)+\frac{p^4}{\beta_m\mathrm{Teich}^m(\lambda)}\right)\ .\end{aligned}\tag{6.19}$$

Curiously, as a consequence of these expressions, there are very simple expressions for the mean and variance of $\nu(\psi)$ as ψ ranges over $\mathbb{F}_p^*$

$$\frac{1}{p-1}\sum_{\psi\in\mathbb{F}_p^*}\nu(\psi) \;=\; p^4+1$$

and

$$\frac{1}{p-1}\sum_{\psi\in\mathbb{F}_p^*}\nu^2(\psi)-\left(\frac{1}{p-1}\sum_{\psi\in\mathbb{F}_p^*}\nu(\psi)\right)^2 \;=\; p^4(p-4)\ .$$

6.3. The number of rational points over $\mathbb{F}_{p^s}^5$ when $5\not| p^s-1$

Given (6.7) and (6.8) it is natural to conjecture that the number of rational points for $x\in\mathbb{F}_{p^s}^5$ will be given by replacing p by $q=p^s$ in these expressions. That is

$$\nu_s(\psi) \;=\; 1+q^4+\sum_{m=1}^{q-2}\beta_{s,m}\,\mathrm{Teich}^m(\lambda) \tag{6.20}$$

with

$$\beta_{s,m} \;=\; \lim_{n\to\infty}\frac{a_{m(1+q+q^2+\ldots+q^{n+1})}}{a_{m(1+q+q^2+\ldots+q^n)}}\ .$$

In order to evaluate the p-adic limit we proceed as before. Now we meet the ratio

$$\frac{a_{rq}}{a_r} \;=\; \frac{a_{rp^s}}{a_{rp^{s-1}}}\frac{a_{rp^{s-1}}}{a_{rp^{s-2}}}\cdots\frac{a_{rp}}{a_r} \;=\; \prod_{\ell=1}^{s}\frac{\Gamma_p(5rp^\ell+1)}{\Gamma_p^5(rp^\ell+1)}$$

with $r=m/(1-q)$. The quantities $A_m\left(mq/(1-q)\right)$ have the same form as in (6.14) but with p replaced by q. Thus we have

$$\beta_{s,m} \;=\; \prod_{\ell=1}^{s}\frac{\Gamma_p\left(\frac{5mp^\ell}{1-q}+1\right)}{\Gamma_p^5\left(\frac{mp^\ell}{1-q}+1\right)}\,\frac{\prod_{j=1}^{5m}\left(\frac{5mq}{1-q}+j\right)}{\prod_{j=1}^{m}\left(\frac{mq}{1-q}+j\right)^5} \tag{6.21}$$

One may check this result numerically for low values of p and s. The computation is easy however the 'experimental data' is difficult to obtain since one is dealing with spaces that have p^{5s} points that defeat the computer already for low values of p and s.

7. Calabi–Yau Manifolds of Zero Dimension

7.1. The rational points of a quadric

In this section we will repeat our analysis for a Calabi–Yau manifold of dimension zero. We shall understand by this the points that solve the quadratic equation

$$P(x) = x_1^2 + x_2^2 - 2\psi x_1 x_2$$

in $\mathbb{F}_p\mathbb{P}^1$. Note also that the three monomials correspond to the points of the unique reflexive polyhedron in one dimension

Over $\mathbb{P}^1$ the situation is, of course, that for $\psi^2 \neq 1$ there are two solutions and these coincide precisely when $\psi^2 = 1$. Over $\mathbb{F}_p\mathbb{P}^1$ the situation is more involved since the solutions that exist over $\mathbb{P}^1$ need not be rational. So the number of solutions over $\mathbb{F}_p\mathbb{P}^1$ may be 0, 1, or 2, depending on p and ψ.

Let us first recall some terminology. For $a \in \mathbb{F}_p^*$ we say that a is a *square residue* in $\mathbb{F}_p$ if it has a square root, that is $a = y^2$ for some $y \in \mathbb{F}_p$. The number a is a *square non-residue* otherwise. For example, in $\mathbb{F}_7$, the numbers $1 = (\pm 1)^2$, $2 = (\pm 3)^2$, and $4 = (\pm 2)^2$ are square residues, whereas 3, 5 and 6 are square non-residues.

The *Legendre symbol* $\left(\frac{a}{p}\right)$ is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1\,, \text{ if } a \text{ is a square residue}\,, \\ -1\,, \text{ if } a \text{ is a non-square residue}\,, \\ 0\,, \text{ if } p|a\,. \end{cases}$$

Let now M_ψ be a Calabi–Yau manifold of zero dimensions as defined above. We denote by $N(\psi)$ the number of solutions of the equation $P = 0$ over $\mathbb{F}_p\mathbb{P}^1$. It is easy to show, by considering the formula for the solution of a quadratic equation, that this number is given by

$$N(\psi) = 1 + \left(\frac{\psi^2 - 1}{p}\right) = \begin{cases} 1, \text{ if } \psi^2 - 1 \equiv 0 \bmod p \\ 2, \text{ if } \psi^2 - 1 \text{ is a non-zero square residue of } p \\ 0, \text{ if } \psi^2 - 1 \text{ is a non-square residue of } p\,. \end{cases} \tag{7.1}$$

Note that this means that $N(\psi)$ depends only on ψ^2-1. For $p=7$, for example, (7.1) yields

$$\begin{aligned} N(\pm 3) &= 2 \quad \text{since} \quad (\pm 3)^2 - 1 = 1, \quad \text{a square residue,} \\ N(\pm 2) &= 0 \quad \text{since} \quad (\pm 2)^2 - 1 = 3, \quad \text{a non-square residue,} \\ N(\pm 1) &= 1 \quad \text{since} \quad (\pm 1)^2 - 1 = 0, \\ N(0) &= 0 \quad \text{since} \quad 0^2 - 1 = 6, \quad \text{a non-square residue.} \end{aligned}$$

One of the objects of this section is to obtain the simple result (7.1) using the methods of the previous sections, *i.e.* in terms of the 'periods' and 'semiperiods' of M_ψ. Because of the simplicity of this example, we will be able to see explicitly how the arithmetic structure is finer than the complex structure.

7.2. Solution in terms of periods

Consider first the periods for M_ψ over the complex numbers and set $\lambda = (2\psi)^{-2}$. The Picard–Fuchs equation may be derived by the methods of §3. We find that

$$\mathcal{L}\,\varpi_0 \;=\; 0 \quad \text{with} \quad \mathcal{L} \;=\; \vartheta - 2\lambda\,(2\vartheta+1) \quad ,$$

is a first order equation that can be solved in closed form. Its solution, corresponding to the fundamental period, is

$$\varpi_0(\lambda) = f_0(\lambda) = (1-4\lambda)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(2n)!}{n!^2}\lambda^n \quad . \tag{7.2}$$

The semiperiods $\varpi_j\,,\ j>1$ are solutions of the equations

$$\vartheta^j \mathcal{L}\varpi_j = 0 \quad .$$

and in particular

$$\varpi_1 \;=\; f_0(\lambda)\log\lambda + f_1(\lambda) \quad \text{with} \quad f_1(\lambda) = 2\,\sum_{n=1}^{\infty} \frac{(2n)!}{n!^2}(\sigma_{2n}-\sigma_n)\;\lambda^n$$

Over the complex numbers, the equation $P(x)=0$, has two different solutions except when $\psi^2=1$, in which case there is only one solution $x_1=\psi x_2$. If $\psi\neq 1$ it may be eliminated from P by a coordinate transformation. The case $\psi^2=1$ is the analog of the

conifold singularity for the quintic in the sense that it corresponds to a point for which both $P = 0$ and $\mathrm{d}P = 0$.

To compute $N(\psi)$ we recall the relation $N(\psi) = \frac{\nu(\psi)-1}{p-1}$ with

$$\nu(\psi) \;=\; \sum_{\mathbb{F}_p^2} \Big(1 - P(x)^{p(p-1)}\Big) + \mathcal{O}\big(p^2\big) \quad . \tag{7.3}$$

These formulas give $\nu(\psi) + \mathcal{O}\big(p^2\big)$ which is actually the exact result since $\nu(\psi) < p^2$. We set $k = \frac{(p-1)}{2}$ and after some straightforward algebra we find:

$$\text{For } \psi = 0 \;:\quad \nu(0) \;=\; (-1)^{k+1} + p\left(1 + (-1)^k\right) = \begin{cases} 2p-1, & \text{if } p \equiv 1 \bmod 4 \\ \\ 1, & \text{if } p \equiv 3 \bmod 4 \end{cases} \tag{7.4}$$

$$\text{For } \psi \neq 0 \;:\quad \nu(\psi) \;=\; -\,{}^{(p-1)}f_0(\lambda^p) - p\,{}^{(p-1)}f_1'(\lambda^p) + 2p + \mathcal{O}\big(p^2\big)$$

and therefore

$$\text{For } \psi = 0 \;:\quad N(0) \;=\; 1 + (-1)^k = \begin{cases} 2, & \text{if } p \equiv 1 \bmod 4 \\ \\ 0, & \text{if } p \equiv 3 \bmod 4 \end{cases} \tag{7.5}$$

$$\text{For } \psi \neq 0 \;:\quad N(\psi) \;=\; (1+p)\left(1 + {}^{(p-1)}f_0(\lambda^p)\right) + p\,{}^{(p-1)}f_1'(\lambda^p) - 2p + \mathcal{O}\big(p^2\big)\,.$$

As before, ${}^{(p-1)}f_j(\lambda^p)$ is the truncation of the series f_j to p terms. Note the difference in signs in (7.4) as compared with the similar result for the quintic three-fold. This difference comes from the factor of $(p-1)^2 = 1+\mathcal{O}(p)$, as opposed to a factor of $(p-1)^5 = -1+\mathcal{O}(p)$ for the quintic threefold, obtained after summing (7.3) over $\mathbb{F}_p^2$. Comparing with (7.1), we obtain interesting identities that relate the periods to the Legendre symbol. For $\psi = 0$, we have the well known result

$$\left(\frac{p-1}{p}\right) = (-1)^{\frac{1}{2}(p-1)} \;,$$

and for $\psi \neq 0$ we have

$$\left(\frac{\psi^2-1}{p}\right) \equiv \;{}^{\frac{1}{2}(p-1)}f_0(\lambda) \mod \mathrm{p}\;.$$

We would like to be able to write down exact p-adic expressions for our results for $\psi \neq 0$. To this end let

$$\beta(\lambda) \;=\; \sum_{m=0}^{p-1} \beta_m \mathrm{Teich}^m(\lambda)\;, \qquad \text{with} \qquad \beta_m \;=\; \frac{\Gamma_p\left(\frac{2mp}{1-p}+1\right)}{\Gamma_p^2\left(\frac{mp}{1-p}+1\right)} \frac{\prod_{j=1}^{2m}\left(\frac{2mp}{1-p}+j\right)}{\prod_{j=1}^{m}\left(\frac{mp}{1-p}+j\right)^2}\;.$$

By means of Gauss sums, one obtains the exact p-adic expression we are looking for

$$N(\psi) + \frac{\beta(\lambda) - 1}{p-1} = 2 \qquad \text{or equivalently,} \qquad \nu(\psi) = 2p - \beta(\lambda) \ .$$

It is also easy to check these relations numerically.

Comparing with our results for the quintic 3-fold, we note again the difference in the sign in front of $\beta(\lambda)$ and the extra term $2p$. The latter is the analog of the extra contributions in $\nu(\psi)$ whenever $5|(p-1)$; for the zero dimensional Calabi–Yau manifold, this term is always present because $2|(p-1)$ for all odd primes.

Comparing with (7.1), we have the following exact relation:

$$\left(\frac{\psi^2 - 1}{p}\right) = \frac{p - \beta(\lambda)}{p-1} \ ,$$

which implies very intriguing relations between the period and the semiperiods:

Condition on ψ	Implication
$\psi^2 = 1$	$\beta(1/4) = \ p$
$\psi^2 - 1$ is a square residue	$\beta(\lambda) = \ 1$
$\psi^2 - 1$ is a non-square residue	$\beta(\lambda) = \ 2p - 1$

These relations imply, in turn, congruence relations for the periods and semiperiods at each order in a p expansion. Examples of these congruences for the truncation of the fundamental period are:

$$\sum_{n=0}^{\frac{1}{2}(p-1)} \frac{(2n)!}{n!^2}\, 2^{-2n} \equiv 0 \mod p \ , \text{ for } \psi^2 = 1 \ ,$$

$$\sum_{n=0}^{\frac{1}{2}(p-1)} \frac{(2n)!}{n!^2}\, \lambda^{-n} \equiv \begin{cases} 1 \mod p \ , \text{ if } \psi^2 - 1 \text{ is a square residue} \ , \\ -1 \mod p \ , \text{ if } \psi^2 - 1 \text{ is a non-square residue} \ . \end{cases}$$

8. CY Manifolds of Dimension One: Elliptic Curves

In this section we consider the case of nonsingular cubics

$$A_\psi : \qquad x^3 + y^3 + z^3 - 3\psi\, xyz = 0 \tag{8.1}$$

for which $\psi^3 \neq 1$. In order to count the number of points of A_ψ over a finite field we will work instead with its mirror (we will be interested later in comparing the number of rational points of the quintic with its mirror), which, in affine coordinates, can be given by

$$B_\lambda : \qquad xy(1-x-y) = \lambda, \qquad \lambda = 1/(3\psi)^3 \ .$$

The monomial change of variables

$$x \mapsto x^2/y, \qquad y \mapsto y^2/x$$

on the torus $x \neq 0, y \neq 0, z = 1$ gives rise to a degree three map between these families. It is not hard to check that B_λ is the quotient of A_ψ by the automorphisms $(x, y) \mapsto (\zeta x, \zeta^{-1} y)$, where $\zeta^3 = 1$.

We may obtain Weierstrass models for our families, for example, using the algorithm in [21]. We find

$$\begin{aligned} A_\psi : &\qquad y^2 - 3\psi xy + 9y = x^3 - 27(\psi^3 + 1) \\ B_\lambda : &\qquad y^2 + xy + \lambda = x^3 \ . \end{aligned}$$

An elliptic curve over $\mathbb{C}$ has a natural group structure and this structure is manifest also when we consider the elliptic curve over a finite field [22]. With respect to this group structure we verify that our previous map is a degree 3 isogeny (an isogeny is a map that preserves the group structure of the elliptic curve); more precisely, B_λ is isomorphic to the quotient of A_ψ by the subgroup of order 3 generated by $\big(3(\psi+1), 9\psi(\psi+1)\big)$.

It is known that isogenous elliptic curves have the same number of points over a given finite field and we will hence restrict ourselves to counting points on B_λ. It is not difficult to check that for $\lambda \neq 0$ there are exactly three points of B_λ not included among those satisfying

$$xy(1-x-y) = \lambda, \qquad x \neq 0, \qquad y \neq 0 \ .$$

(We may note, for example, that each side of the Newton polygon of this equation has integral length 1, corresponding to a single rational point on the appropriate non-singular toric surface.)

For $\lambda \in \mathbb{F}_q^*$ let $N^*(\lambda)$ be the number of projective solutions of (8.1) with $x, y \in \mathbb{F}_q^*$. By the above discussion the number of points $N(\lambda)$ of B_λ over $\mathbb{F}_q$, and also of A_ψ if there is a $\psi \in \mathbb{F}_q$ with $\lambda = 1/(3\psi)^3$, is $N^*(\lambda) + 3$.

For any $x \in \mathbb{F}_q^*$ we have

$$\sum_\chi \chi(x) = \begin{cases} q-1 & x = 1 \\ 0 & \text{otherwise} \end{cases},$$

where the sum is over all multiplicative characters of $\mathbb{F}_q^*$. Therefore, for $\lambda \in \mathbb{F}_q^*$

$$N^*(\lambda) = \frac{1}{q-1} \sum_\chi \chi(\lambda^{-1}) \sum_{\substack{x_i \in \mathbb{F}_q^* \\ x_1+x_2+x_3=1}} \chi(x_1 x_2 x_3) \ .$$

Let us recall the definition of the *Jacobi sum* associated to characters $\chi_1, \ldots, \chi_d$ of $\mathbb{F}_q^*$

$$J(\chi_1, \ldots, \chi_d) \ = \sum_{\substack{x_i \in \mathbb{F}_q \\ x_1+\cdots+x_d=1}} \chi_1(x_1) \cdots \chi_d(x_d) \ ,$$

where we follow the standard convention

$$\chi(0) = \begin{cases} 1 & \text{if } \chi = \mathbf{1} \\ 0 & \text{otherwise} \end{cases},$$

where, in this context, $\mathbf{1}$ denotes the trivial character. For standard facts relating to Jacobi sums see, for example, [23].

The inner sum in (1.2) is almost the Jacobi sum $J(\chi, \chi, \chi)$ except for the terms where one of the x_i's is zero. Such a term contributes to the Jacobi sum only if χ is trivial, in which case it contributes a 1. By the principle of inclusion/exclusion the neglected terms amount to $3q - 3$ (q for each of the three cases where one x_i is zero, 1 for each of the three cases where two of them are zero). Hence,

$$N^*(\lambda) = \frac{1}{q-1} \sum_\chi \chi(\lambda^{-1}) J(\chi, \chi, \chi) - 3 \,,$$

and

$$N(\lambda) = \frac{1}{q-1} \sum_\chi \chi(\lambda^{-1}) J(\chi, \chi, \chi) \,. \tag{8.2}$$

Using the Gross-Koblitz formula (which we will come to in the following section) we may write $J(\chi, \chi, \chi)$ in terms of Γ_p and is not hard to verify then that we recover the analogue of our previous formula for $N(\lambda)$.

9. The Relation to Gauss Sums

9.1. Dwork's character

We record in this section expressions for the $\nu(\psi)$ in terms of Gauss sums. We begin by reviewing the properties of Dwork's character $\Theta : \mathbb{F}_p \to \mathbb{C}_p$; a full account may be found in [24,10]. The character is defined by

$$\Theta(x) \;=\; F(\pi\,\mathrm{Teich}(x))$$

with F the function given in (6.12) and π a number in $\mathbb{C}_p$ such that $\pi^{p-1} = -p$.

The introduction of π merits some comment. Since the p-adic order of π is $1/(p-1)$, which is not an integer, π is not a p-adic number but lies in an extension of $\mathbb{Q}_p$. π arises in connection with finding nontrivial q'th roots of unity which is clearly closely allied to the question of finding multiplicative characters. It can be shown that all the roots of the equation $X^q - 1 = 0$ lie in $\mathbb{Q}_p(\pi)$, the field of elements of the form $\sum_{r=0}^{p-2} a_r\,\pi^r\ ,\ a_j \in \mathbb{Q}_p$. A second fact is that the series $\exp(X)$ is less convergent in the p-adic case than in classical analysis, owing to the presence of the factorials in the denominators. It is straightforward to see that it converges in the disk $|X|_p < |\pi|_p$. It is perhaps natural therefore to consider the series $\exp(\pi Y)$ which converges in the disk $|Y|_p < 1$.

Now

$$F(\pi X) \;=\; \exp\bigl(\pi(X - X^p)\bigr) \;=\; \sum_{n=0}^{\infty} c_n\,(\pi X)^n \tag{9.1}$$

and it is an essential fact that the series on the right converges for $|X|_p < 1 + \epsilon$ for a fixed positive number ϵ.

There is a trap here for the unwary which is to assume that since a Teichmüller representative satisfies the equation $X^p = X$ that $\Theta(x) = \exp(0) = 1$. This turns out to be false owing to the fact that the X-disk that ensures that $|X - X^p|_p < 1$ is $|X|_p < 1$. There is thus an issue of analytic continuation as discussed in §2.2. We are to regard $F(\pi X)$ and hence Θ as defined by the series on the right of (9.1). $F(\pi X)$ then equals $\exp\bigl(\pi(X - X^p)\bigr)$ for $|X|_p < 1$ but not for $|X|_p \geq 1$. From the series expansion we have

$$\Theta(1) \;=\; 1 + \pi + \mathcal{O}\bigl(\pi^2\bigr)$$

from which it follows that $\Theta(1) \neq 1$. On the other hand if $X^p = X$ then

$$F(\pi X)^p \;=\; \exp(p\pi X - p\pi X^p) \;=\; 1$$

since the presence of the p in the exponent ensures the convergence of the series. In particular $\Theta(1)$ is a p'th root of unity and since $\Theta(x) = 1 + \pi\,\mathrm{Teich}(x) + \mathcal{O}\big(\pi^2\big)$ we see also that $\Theta(x) = \Theta(1)^{\mathrm{Teich}(x)} = \Theta(1)^x$.

Since for $x, y \in \mathbb{F}_p$

$$\mathrm{Teich}(x+y) \;=\; \mathrm{Teich}(x) + \mathrm{Teich}(y) + pZ$$

for some $Z \in \mathbb{Z}_p$ it follows that Θ is a nontrivial additive character, that is

$$\Theta(x+y) \;=\; \Theta(x)\,\Theta(y)\;.$$

Dwork also adapted this construction to give a additive character $\Theta_s : \mathbb{F}_q \to \mathbb{C}_p$ for $q = p^s$

$$\Theta_s(x) \;=\; \Theta(\mathrm{tr}(x))$$

where $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is the trace map

$$\mathrm{tr}(x) \;=\; x + x^p + x^{p^2} + \cdots + x^{p^{s-1}}$$

Now the limit $\mathrm{Teich}(x) = \lim_{n\to\infty} x^{q^n}$ exists also for $x \in \mathbb{F}_q$ though $\mathrm{Teich}(x)$, for $x \neq 0$, is now a unit (has unit norm) of $\mathbb{C}_p$ but is not in general in $\mathbb{Z}_p$. We have

$$\mathrm{Teich}\left(\mathrm{tr}(x)\right) \;=\; \sum_{\ell=0}^{s-1} \mathrm{Teich}^{p^\ell}(x) + pZ$$

with Z an integer of $\mathbb{C}_p$. It follows that

$$\Theta_s(x) \;=\; \Theta(1)^{\sum_{\ell=0}^{s-1} \mathrm{Teich}^{p^\ell}(x)}\;.$$

Note however that we cannot, in general, write the right hand side of this relation as the product

$$\prod_{\ell=0}^{s-1} \Theta(1)^{\mathrm{Teich}^{p^\ell}(x)}$$

since the $\mathrm{Teich}^{p^\ell}(x)$ are not in general in $\mathbb{Z}_p$. Dwork showed nevertheless that Θ_s has the remarkable splitting property

$$\Theta_s(x) \;=\; \prod_{\ell=0}^{s-1} \Theta\Big(x^{p^\ell}\Big)\;.$$

9.2. Evaluation of $\nu(\psi)$ in terms of Gauss sums

It is useful to define 'Gauss sums'

$$g_n(y) \;=\; \sum_{x\in\mathbb{F}_p^*} \Theta(yx^5)\,\mathrm{Teich}^n(x) \quad \text{and} \quad G_n \;=\; \sum_{x\in\mathbb{F}_p^*} \Theta(x)\,\mathrm{Teich}^n(x)\ . \tag{9.2}$$

G_n is a genuine Gauss sum, that is a sum of an additive character, Θ, times a multiplicative character, $\mathrm{Teich}^n(x)$. Note however that our convention is nonstandard with respect to G_0. The standard convention for the Gauss sum for the case that the multiplicative character is the identity takes the sum to be over $\mathbb{F}_p$ instead of $\mathbb{F}_p^*$ with the result that the Gauss sum is zero in this case. We find it convenient here to define G_0 as above with the result that $G_0 = \sum_{x\in\mathbb{F}_p^*}\Theta(x) = -1$.

Gauss sums are the analogues for finite fields of the classical Γ-function

$$\Gamma(s) \;=\; \int_0^\infty \frac{dt}{t}\, t^s e^{-t}$$

which is an integral, with multiplicative Haar measure dt/t, of a product of the additive character e^{-t} and the multiplicative character t^s. In our case it is a consequence of standard properties of Gauss sums [23,25] that

$$G_n\, G_{-n} \;=\; (-1)^n\, p \quad \text{for} \quad p-1 \not| \, n \tag{9.3}$$

which is an analogue for finite fields of the reflection formulae (2.10) and (2.9). An explicit expression for G_n for the the case $p-1\not| \, n$ is provided by the Gross-Koblitz formula (see, for example, [25]; the original reference is [26])

$$G_n \;=\; p\,(-p)^{-\left\langle \frac{n}{p-1}\right\rangle}\,\Gamma_p\left(1-\left\langle\frac{n}{p-1}\right\rangle\right)\ ;\ \ p-1\not| \, n\ . \tag{9.4}$$

We will make repeated use in the following of an identity. Let t be an integer and let t' be the reduction of t mod $p-1$ to the range $0\le t'\le p-2$ then by considering the definition (9.2) for G_{-t} and expanding $\Theta(x)$ as a series we find

$$(p-1)\sum_{r=0}^{\infty} c_{r(p-1)+t'}\,\pi^{t'}\,(-p)^r \;=\; G_{-t}$$

and by returning with this identity to the series expansion of Θ we see that

$$\Theta(x) \;=\; \frac{1}{p-1}\sum_{m=0}^{p-2} G_{-m}\,\mathrm{Teich}^m(x)\;.$$

Now we write

$$\begin{aligned}\Theta(yP(x)) \;&=\; \left(\prod_{i=1}^{5}\Theta(yx_i^5)\right)\Theta\left(-5\psi y\prod_{i=1}^{5}x_i\right)\\ &=\; \frac{1}{p-1}\sum_{m=0}^{p-2} G_{-m}\,\mathrm{Teich}^m(-5\psi y)\prod_{i=1}^{5}\Theta(yx_i^5)\,\mathrm{Teich}^m(x_i)\;.\end{aligned}\tag{9.5}$$

Since Θ is a character we have

$$\sum_{y\in\mathbb{F}_p}\Theta(yP(x)) \;=\; \begin{cases} p\;, & \text{if } P(x)=0\\ 0\;, & \text{if } P(x)\neq 0\;.\end{cases}\tag{9.6}$$

so if we sum this relation for $x\in(\mathbb{F}_p^*)^5$ we have

$$p\nu^*(\psi)-(p-1)^5 \;=\; \frac{1}{p-1}\sum_{m=0}^{p-2} G_{-m}\,\mathrm{Teich}^m(-5\psi)\sum_{y\in\mathbb{F}_p^*} g_m^5(y)\,\mathrm{Teich}^m(y)\;.\tag{9.7}$$

Consider now the case that $5\nmid(p-1)$ and choose ℓ such that $5\ell=1 \bmod p-1$. We have

$$\sum_{y\in\mathbb{F}_p^*} g_m^5(y)\,\mathrm{Teich}^m(y) \;=\; \sum_{y\in\mathbb{F}_p^*}\left(\sum_{x\in\mathbb{F}_p^*}\Theta(yx^5)\,\mathrm{Teich}^{lm}(yx^5)\right)^5 \;=\; (p-1)G_{lm}^5\;.$$

If we use this identity in (9.7), choose ℓm as the variable of summation and also use expressions from §4.1 to relate $\nu^*(\psi)$ to $\nu(\psi)$ we find

$$\nu(\psi) \;=\; 1+p^4+\sum_{m=1}^{p-2}\frac{G_m^5}{G_{5m}}\,\mathrm{Teich}^{-m}(\lambda)\;.\tag{9.8}$$

Comparing this to our expression for $\nu(\psi)$ in §6 we see that $\beta_{p-1-m}=\frac{G_m^5}{G_{5m}}$ and the identity that relates β_{p-1-m} to β_m is now a simple consequence of(9.3):

$$\beta_{p-1-m} \;=\; \frac{G_m^5}{G_{5m}} \;=\; p^4\,\frac{G_{5m}}{G_m^5} \;=\; \frac{p^4}{\beta_m}\;.$$

By using the Gross-Koblitz formula (9.4) in (9.8) we recover (and establish) (6.17).

9.3. The case $5|p-1$

We return now to (9.7) and note that when $5|(p-1)$ there are nontrivial fifth roots of unity. If ζ is such a root then by replacing x by $x\zeta$ in the first of (9.2) we see that $g_m(y) = 0$ unless $5|m$. Thus for $\psi \neq 0$

$$\begin{aligned} p\nu^*(\psi) - (p-1)^5 &= \frac{1}{p-1}\sum_{m=0}^{k-1}(-1)^m G_{-5m}\mathrm{Teich}^{-m}(\lambda)\sum_{y\in\mathbb{F}_p^*} g_{5m}^5(y)\,\mathrm{Teich}^{5m}(y) \\ &= \frac{1}{p-1}\sum_{m=0}^{k-1}(-1)^m G_{5m}\mathrm{Teich}^{m}(\lambda)\sum_{y\in\mathbb{F}_p^*} g_{-5m}^5(y)\,\mathrm{Teich}^{-5m}(y) \end{aligned}$$

.

Now by again expanding the character as a series we have

$$g_{-5m}(y)\,\mathrm{Teich}^{-m}(y) = \frac{1}{p-1}\sum_{n=0}^{p-2} G_{-n}\sum_{x\in\mathbb{F}_p^*}\mathrm{Teich}^{n-m}(yx^5) = \sum_{s=0}^{4} G_{-(m+ks)}\,\mathrm{Teich}^{ks}(y) \tag{9.9}$$

In this way we find

$$p\nu^*(\psi) - (p-1)^5 = \sum_{\substack{s_i=0 \\ 5|\sum s_i}}^{4}\sum_{m=0}^{k-1}(-1)^m\mathrm{Teich}^m(\lambda)\,G_{5m}\prod_{j=1}^{5} G_{-(m+ks_j)}\ . \tag{9.10}$$

In this expression it is natural to think of the vectors $\mathbf{s} = (s_1, \ldots, s_5)$ as corresponding to monomials $x^{\mathbf{s}}$ of degree $5w$, $0 \leq w \leq 4$ that are subject to the constraint $s_i \leq 4$. Let us call the set of such monomials $\mathcal{S}$ and consider the operation, $\varphi:\ \mathcal{S} \to \mathcal{S}$, of adding $\mathbf{1}$ to $\mathbf{s}$ and then reducing the result mod 5. There is no $\mathbf{s}$ invariant under φ, and φ^5 is the identity so the monomials fall into orbits of φ of length 5. Consider now the contribution of such an orbit to the sum (9.10). The contribution of $\varphi(\mathbf{s})$ is obtained by making the replacement $m \to m+k$ in the term corresponding to $\mathbf{s}$ so we may sum over the orbit by choosing a representative and increasing the range of m to $0 \leq m \leq p-2$. Since two monomials which differ merely by a permutation of their components contribute equally to the sum we may as well identify such monomials. If we do this then it is no longer true that no monomial is invariant since now $\varphi(4,3,2,1,0) \simeq (4,3,2,1,0)$. It is simple to enumerate all the orbits and assign representative monomials which we choose to be of lowest degree. This process

is unambiguous apart from the orbit of the monomial $(4,1,0,0,0)$ which contains also $(2,1,1,1,0)$. In this case we choose $(4,1,0,0,0)$ to represent the orbit. As a consequence, and as was to be anticipated from our discussion of §5 the monomial $(2,1,1,1,0)$, which is interior to a codimension one face of Δ, does not occur explicitly in our list. Which we present in Table 9.1 below

monomial	degree	$\gamma_{\mathbf{v}}$	length of orbit
(0,0,0,0,0)	0	1	5
(4,1,0,0,0)	5	20	5
(3,2,0,0,0)	5	20	5
(3,1,1,0,0)	5	30	5
(2,2,1,0,0)	5	30	5
(4,3,2,1,0)	10	24	1

Table 9.1

Incorporating these considerations brings us to the expression

$$p\nu^*(\psi) - (p-1)^5 \;=\; \sum_{\mathbf{v}} \gamma_{\mathbf{v}} \sum_{m=0}^{p-2} (-1)^m \mathrm{Teich}^m(\lambda)\, G_{5m} \prod_{j=1}^{5} G_{-(m+kv_j)} \tag{9.11}$$

with coefficients $\gamma_{\mathbf{v}}$, given in the table, that account for the permutations of the components of the monomials $\mathbf{v}$.

The quantity $\nu^*(\psi)$ differs from $\nu(\psi)$ by the constant ν_Y but this is given by a more complicated expression than was the case for $5 \nmid p-1$. In order to calculate ν_Y as the difference $\nu(0) - \nu^*(0)$ we return to the first of equations (9.5) from which it follows that

$$p\nu(0) \;=\; p^5 + \sum_{y \in \mathbb{F}_p^*} \left(g_0(y)+1\right)^5 \quad \text{and} \quad p\nu^*(0) \;=\; (p-1)^5 + \sum_{y \in \mathbb{F}_p^*} g_0^5(y)\ .$$

now by specialising (9.9) to the case $m = 0$ we have

$$g_0(y) \;=\; \sum_{s=0}^{4} G_{-ks}\, y^{ks} \quad \text{and} \quad g_0(y)+1 \;=\; \sum_{s=1}^{4} G_{-ks}\, y^{ks}\ .$$

and from this we obtain the useful expression

$$p\big(\nu(\psi)-\nu^*(\psi)\big) \;=\; p^5-(p-1)^5-(p-1)\sum_{\substack{s_i\in\mathcal{S}\\ \prod_i s_i=0}}\prod_{i=1}^{5}G_{-ks_i} \tag{9.12}$$

with the sum on the right being over vectors of $\mathcal{S}$ which have at least one component that is zero. This expression will guide us in choosing how best to write the coefficients that appear in the expansion

$$\nu(\psi) \;=\; p^4+\sum_{\mathbf{v}}\gamma_{\mathbf{v}}\sum_{m=0}^{p-2}\beta_{\mathbf{v},m}\mathrm{Teich}^m(\lambda)\;. \tag{9.13}$$

This question is more involved for the present case that $5|p-1$ since there are now five constant terms on the right hand side, those with $m=ks,\ 0\le s\le 4$, and the question arises as to how best to distribute the constant that we have to transfer to the right hand side when we pass from $\nu^*(\psi)$ to $\nu(\psi)$. For $m\neq ks$ the coefficients are not affected by this constant and a little algebra, making use of (6.18) and the Gross-Koblitz formula yields *for the case* $k\not| m$ the relations

$$\begin{aligned}
\beta_{\mathbf{v},m} \;&=\; \frac{(-1)^m}{p}\,G_{5m}\prod_{j=1}^{5}G_{-(m+kv_j)}\\
&=\; p^4\,\frac{G_{5m}}{\prod_{i=1}^{5}G_{m+kv_i}}\\
&=\; (-p)^{w(\mathbf{v})}\,\frac{\mathfrak{G}_{5m}}{\prod_{i=1}^{5}\mathfrak{G}_{m+kv_i}}\\
&=\; (-p)^{w_m(\mathbf{v})}\,\frac{\Gamma_p\left(1-\left\langle\frac{5m}{p-1}\right\rangle\right)}{\prod_{i=1}^{5}\Gamma_p\left(1-\left\langle\frac{m+kv_i}{p-1}\right\rangle\right)}
\end{aligned} \tag{9.14}$$

where

$$w_m(\mathbf{v}) \;=\; \sum_{i=1}^{5}\left\langle\frac{\left[\frac{m}{k}\right]+v_i}{5}\right\rangle \quad\text{and}\quad w(\mathbf{v})=w_0(\mathbf{v})=\frac{1}{5}\sum_{i=1}^{5}v_i$$

are the weights of $\mathbf{v}+[m/k]\,\mathbf{1}$ and of $\mathbf{v}$ and we have defined

$$\mathfrak{G}_t \;=\; \Gamma_p\left(1+\frac{tp}{1-p}\right)\prod_{j=1}^{t}\left(j+\frac{tp}{1-p}\right)\;.$$

The point to be made by (9.14) is that these four expressions for $\beta_{\mathbf{v},m}$ are all the same when $k \not| m$ but are all different when $k|m$. If however we compare the desired expression (9.13) with (9.11) and (9.12) we see that we can choose the $\beta_{\mathbf{v},ak}$ such that

$$p\sum_{\mathbf{v}}\gamma_{\mathbf{v}}\sum_{a=0}^{4}\beta_{\mathbf{v},ak} \;=\; -\sum_{\mathbf{v}}\gamma_{\mathbf{v}}\left\{(p-1)\sum_{\substack{a=0\\ z(\mathbf{v}+a\mathbf{1})>0}}^{4}\prod_{i=1}^{5}G_{-(v_i+a)k}+\sum_{a=0}^{4}\prod_{i=1}^{5}G_{-(v_i+a)k}\right\}$$

where $z(\mathbf{u})$ is the function that counts the number of components of a vector $\mathbf{u}$ that are zero mod 5. We therefore choose

$$\beta_{\mathbf{v},ak} \;=\; -\frac{p^{5-z-\delta(z)}}{\prod_{i=1}^{5}G_{(v_i+a)k}} \;=\; -\frac{p^{-\delta(z)}\,(-p)^{w_{ak}(\mathbf{v})}}{\prod_{i=1}^{5}\Gamma_p\left(1-\left\langle\frac{v_i+a}{5}\right\rangle\right)}$$

with z denoting $z(\mathbf{v}+a\mathbf{1})$.

We recognise, in (9.13), the terms corresponding to $(0,0,0,0,0)$ as corresponding to the analog of our previous expression for the case $5 \not| p-1$. The terms corresponding to monomials of degree 5 yield the prolongations of the 'extra terms' that we found in (5.1). The degree 10 monomial is however new and we pause to examine its contribution. For reasons that will become apparent we shall refer to this as the conifold term and denote it by $\mathrm{cnf}(\psi)$. A little simplification quickly reveals that

$$\mathrm{cnf}(\psi) \;=\; 120\,p^2\sum_{m=0}^{k-1}\frac{\prod_{i=0}^{4}\Gamma_p\left(\frac{\frac{m}{k}+i}{5}\right)}{\Gamma_p\left(\frac{m}{k}\right)}\,\mathrm{Teich}^m(\lambda)\;. \tag{9.15}$$

We recognise the combination of Γ_p-functions that appears in the sum as corresponding to the multiplication formula (2.11) and we come to some bookwork [27] that we cannot resist recalling. We need the multiplication formula for $n=5$ and when in addition $5|p-1$ the product $\prod_{i=1}^{4}\Gamma_p\left(\frac{i}{5}\right)=1$ as we see by using (2.9) and noting that $L(1/5)=p-k$ and $L(2/5)=p-2k$. Thus we have

$$\frac{\prod_{i=0}^{4}\Gamma_p\left(\frac{Z+i}{5}\right)}{\Gamma_p(Z)} \;=\; 5^{1-L(Z)}\left(5^{-(p-1)}\right)^{Z_1}\;. \tag{9.16}$$

In this formula we take

$$Z \;=\; \frac{m}{k} \;=\; \frac{5m}{p-1}$$

so we have $L(Z)=p-5m$ and

$$Z_1 \;=\; \frac{1}{p}\left(Z-L(Z)\right) \;=\; \frac{5m-p+1}{p-1} \;=\; \frac{1-L(Z)}{p-1}\;.$$

Let us denote the right hand side of the multiplication formula by t_m. As remarked previously $5^{-(p-1)} = 1 + \mathcal{O}(p)$ so $\left(5^{-(p-1)}\right)^{Z_1}$ is also $1 + \mathcal{O}(p)$. Thus we see that

$$t_m \;=\; 5^{5m} + \mathcal{O}(p) \;.$$

Now following from our observation concerning Z_1 we have

$$t_m = 5^{1-L(Z)} \left(5^{-(p-1)}\right)^{\frac{1-L(Z)}{p-1}}$$

It is perhaps tempting to rearrange the exponents on the right and conclude, falsely, that $t_m = 1$. This is another trap for the unwary since the term $\left(5^{-(p-1)}\right)^{Z_1}$ has to be understood as the binomial series of a quantity that is $1 + \mathcal{O}(p)$. However if we consider t_m^{p-1} we have

$$t_m^{p-1} \;=\; 5^{(p-1)(1-L(Z))} \left(5^{-(p-1)}\right)^{1-L(Z)} \;=\; 1$$

and the cancellation is now valid. Thus it has been shown that $t_m \;=\; 5^{5m} + \mathcal{O}(p)$ and that $t_m^{p-1} = 1$. In other words

$$t_m \;=\; \text{Teich}(5^{5m}) \;.$$

If we now use this result to simplify (9.15) we find, in an obvious notation, that

$$\text{cnf}(\psi) \;=\; 24p^2(p-1)\,\delta\big(\text{Teich}^5(\psi) - 1\big) \;.$$

9.4. Relation to the periods

In this subsection we rewrite the expression for the number of points (9.13) in terms of the periods of the holomorphic $(3,0)$ form Ω. To do this we begin by defining

$$\tilde{\beta}_{\mathbf{v},m} \;=\; (-p)^{w(\mathbf{v})} \frac{\Gamma_p\left(1 + \frac{5mp}{1-p}\right) \prod_{j=1}^{5m} \left(j + \frac{5mp}{1-p}\right)}{\prod_{i=1}^{5} \Gamma_p\left(1 + \frac{(m+kv_i)p}{1-p}\right) \prod_{j=1}^{m+kv_i} \left(j + \frac{(m+kv_i)p}{1-p}\right)} \;. \tag{9.17}$$

These coincide with $\beta_{\mathbf{v},m}$ in (9.14) when $k \nmid m$. In terms of $\tilde{\beta}_{\mathbf{v},m}$ the number of points can be rewritten as

$$\nu(\psi) \;=\; p^4 + \mathcal{I} + \sum_{\mathbf{v}} \gamma_{\mathbf{v}} \sum_{m=0}^{p-2} \tilde{\beta}_{\mathbf{v},m} \text{Teich}^m(\lambda) \;,$$

where the term $\mathcal{I}$ is independent of λ

$$\mathcal{I} \;=\; 30(p-1)p^2 \left(\frac{1}{\Gamma_p\left(\frac{1}{5}\right)^2 \Gamma_p\left(\frac{3}{5}\right)} + \frac{1}{\Gamma_p\left(\frac{1}{5}\right) \Gamma_p\left(\frac{2}{5}\right)^2} \right) \;.$$

We will discuss the interpretation of this constant in [9]. By writing

$$\frac{(m+kv_i)p}{1-p} = \frac{mp}{1-p} - \frac{v_i}{5} - kv_i \ ,$$

it is easy to see that $\tilde{\beta}_{\mathbf{v},m}$ can be written as

$$\tilde{\beta}_{\mathbf{v},m} \ = \ (-p)^{w(\mathbf{v})} \, \frac{\Gamma_p\left(1+\frac{5mp}{1-p}\right) \prod_{j=1}^{5m}\left(j+\frac{5mp}{1-p}\right)}{\prod_{i=1}^{5}\Gamma_p\left(1+\frac{mp}{1-p}-\frac{v_i}{5}\right)\prod_{j=1}^{m}\left(j+\frac{mp}{1-p}-\frac{v_i}{5}\right)} \quad .$$

Recalling (3.10)

$$A_{\mathbf{v},n}(\varepsilon) \ = \ \frac{\Gamma(5n+5\varepsilon+1)}{\Gamma(5\varepsilon+1)} \prod_{i=1}^{5} \frac{\Gamma\left(\varepsilon+1-\frac{v_i}{5}\right)}{\Gamma\left(n+\varepsilon+1-\frac{v_i}{5}\right)} \ ,$$

and writing $a_{\mathbf{v},m} = A_{\mathbf{v},m}(0)$, we have

$$\tilde{\beta}_{\mathbf{v},m} \ = \ -\frac{(-p)^{w(\mathbf{v})}}{\prod_{i=1}^{5}\Gamma_p(1-\frac{v_i}{5})} \lim_{l\to\infty} \frac{a_{\mathbf{v},mp(1+p+\ldots+p^l)}}{a_{\mathbf{v},m(1+p+\ldots+p^l)}} \, A_{\mathbf{v},m}(mp(1+p+\ldots+p^l)) \ .$$

Therefore, by using the identity

$$a_{\mathbf{v},rp+s} = a_{\mathbf{v},rp} \, A_{\mathbf{v},s}(rp) \ ,$$

we see that

$$\tilde{\beta}_{\mathbf{v},m} \ = \ -\frac{(-p)^{w(\mathbf{v})}}{\prod_{i=1}^{5}\Gamma_p(1-\frac{v_i}{5})} \lim_{l\to\infty} \frac{a_{\mathbf{v},m(1+p+\ldots+p^{l+1})}}{a_{\mathbf{v},m(1+p+\ldots+p^l)}} \ .$$

This expression for the coefficients $\tilde{\beta}_{\mathbf{v},m}$ generalizes the one we had obtained for the case $\mathbf{v} = (0,0,0,0,0)$ (see (6.8)).

We now follow the method of Frobenius as in Section (6.2). It will be convenient to define, in analogy with (6.5), the quantity

$$h_{\mathbf{v}}(rp) \ = \ \frac{a_{\mathbf{v},rp}}{a_{\mathbf{v},r}} \ = \ -\Gamma_p(1+5rp) \prod_{i=1}^{5} \frac{\Gamma_p(1-\frac{v_i}{5})}{\Gamma_p(1-\frac{v_i}{5}+rp)} \ ,$$

and to rewrite $\tilde{\beta}_{\mathbf{v},m}$ in terms of these

$$\tilde{\beta}_{\mathbf{v},m} \ = \ -\frac{(-p)^{w(\mathbf{v})}}{\prod_{i=1}^{5}\Gamma_p(1-\frac{v_i}{5})} \, h_{\mathbf{v}}\left(\frac{mp}{1-p}\right) \, A_{\mathbf{v},m}\left(\frac{mp}{1-p}\right) \quad .$$

For each $\mathbf{v}$, a basis of periods and semiperiods is obtained from

$$F_{\mathbf{v}}(\lambda,\epsilon) \;=\; \sum_{m=0}^{\infty} A_{\mathbf{v},m}(\epsilon)\,\lambda^{m+\epsilon}\ ,$$

as

$$\varpi_{\mathbf{v},k}(\lambda) \;=\; \frac{d^{(k)}F_{\mathbf{v}}(\lambda,\epsilon)}{d\epsilon^k}\bigg|_{\epsilon=0} \;=\; \sum_{i=0}^{k}\binom{k}{i} f_{\mathbf{v},i}(\lambda)\log^{k-i}\lambda$$

where the

$$f_{\mathbf{v},i}(\lambda) \;=\; \sum_{m=0}^{\infty}\ \frac{d^i A_{\mathbf{v},m}(\epsilon)}{d\epsilon^i}\bigg|_{\epsilon=0}\ \lambda^m\ ,$$

are regular at $\lambda=0$. We work mod ϵ^5 as before and expand

$$A_{\mathbf{v},m}(\epsilon) = \sum_{k=0}^{4}\frac{1}{k!}\,a_{\mathbf{v},m}\,b_{\mathbf{v},m,k}\ \epsilon^k\ ,$$

$$h_{\mathbf{v}}(\epsilon) = \sum_{k=0}^{4}\frac{1}{k!}\,h_{\mathbf{v}}(0)\,g_{\mathbf{v},k}\ \epsilon^k\ .$$

Noting that $h_{\mathbf{v}}(0)=1$ and that $g_{\mathbf{v},0}=1$ we obtain for the number of points in terms of the periods

$$\nu(\psi) = p^4+\mathcal{I}+\sum_{\mathbf{v}}\gamma_{\mathbf{v}}(-p)^{w(\mathbf{v})}\ \sum_{k=0}^{4}\frac{1}{k!}\left(\frac{p}{1-p}\right)^k g_{\mathbf{v},k}\sum_{j=0}^{4}\frac{1}{j!}\left(\frac{p}{1-p}\right)^j\ {}^{(p-2)}f_{\mathbf{v},j}^{(k)}(\mathrm{Teich}(\lambda))\ ,$$

where, as before, ${}^{(p-2)}f_{\mathbf{v},j}^{(k)}$ refers to the truncated series up to the $(p-2)$-th term.

9.5. The calculation for finer fields

The calculation for the field with $q=p^s$ points proceeds in complete analogy to the calculation presented above. It suffices to replace Θ by Θ_s, p by q, and G_n and $g_n(y)$ by $G_{s,n}$ and $g_{s,n}(y)$. Where

$$g_{s,n}(y) \;=\; \sum_{x\in\mathbb{F}_q^*}\Theta_s(yx^5)\,\mathrm{Teich}^n(x) \quad\text{and}\quad G_{s,n} \;=\; \sum_{x\in\mathbb{F}_q^*}\Theta_s(x)\,\mathrm{Teich}^n(x)\ . \tag{9.18}$$

The only expressions that change their form are the explicit forms of the coefficients in the expansion of the character which are no longer given by (6.13) and the Gross-Koblitz formula which is now

$$G_{s,n} \;=\; (-1)^{s+1}q\,\pi^{-S(n)}\prod_{\ell=0}^{s-1}\Gamma_p\left(1-\left\langle\frac{p^\ell n}{q-1}\right\rangle\right)\ ;\ \ 1\le n\le q-2\ .$$

The important point is that apart from the change to the Gross-Koblitz formula the direct analogs of our expressions from (9.3) on are true. For the case $5 \not| \, q-1$ we check from these expressions that $\nu_s(\psi)$ is as we proposed in (6.20) and (6.21). For the case $5|q-1$ we have an expression analogous to (9.13)

$$\nu_s(\psi) \;=\; q^4 + \sum_{\mathbf{v}} \gamma_{\mathbf{v}} \sum_{m=0}^{q-2} \beta_{s,\mathbf{v},m} \, \mathrm{Teich}^m(\lambda) \; . \tag{9.19}$$

with

$$\beta_{s,\mathbf{v},m} \;=\; \begin{cases} q^4 \, \dfrac{G_{s,5m}}{\prod_{i=1}^5 G_{s,(m+v_i k)}} \; , & k \not| \, m \\ \\ -\dfrac{q^{5-z-\delta(z)}}{\prod_{i=1}^5 G_{s,(v_i+a)k}} \; , & k|m \; . \end{cases}$$

9.6. A CY hypersurface in a toric variety

For a general Calabi–Yau hypersurface in a toric variety we return to the notation and conventions of §9.3. We have

$$p\nu^*(\alpha) - (p-1)^{N+4} \;=\; \sum_{y,x_i \in \mathbb{F}_p^*} \Theta\!\left(\sum_{\mathbf{m}} \alpha_{\mathbf{m}} x^{\mathbf{m}} - \phi \, Q \right) \;=\; \sum_{y,x_i \in \mathbb{F}_p^*} \Theta(-\phi \, Q) \prod_{\mathbf{m}} \Theta(\alpha_{\mathbf{m}} x^{\mathbf{m}}) \; .$$

By expanding the characters as a series we find

$$\begin{aligned} p\nu^*(\alpha) - (p-1)^{N+4} \;=\; & \frac{1}{(p-1)^{\mathrm{pts}(\Delta)}} \sum_{s=0}^{p-2} (-\phi)^{-s} \, G_s \prod_{\mathbf{m}} \sum_{s_{\mathbf{m}}=0}^{p-2} \alpha_{\mathbf{m}}^{s_{\mathbf{m}}} \, G_{-s_{\mathbf{m}}} \times \\ & \times \sum_{y \in \mathbb{F}_p^*} y^{\sum_{\mathbf{m}} s_{\mathbf{m}} - s} \sum_{x_i \in \mathbb{F}_p^*} x^{\sum_{\mathbf{m}} s_{\mathbf{m}} \, \mathbf{m} - s\mathbf{1}} \; , \end{aligned}$$

where $\mathrm{pts}(\Delta)$ denotes the number of monomials of Δ. The x and y sums impose the conditions

$$\sum_{\mathbf{m}} s_{\mathbf{m}} \, \mathbf{m} \;\equiv\; s\mathbf{1} \qquad \text{and} \qquad \sum_{\mathbf{m}} s_{\mathbf{m}} \;\equiv\; s \qquad \mathrm{mod}\ (p-1) \; .$$

The second of these is redundant unless all the multidegrees of P divide $p-1$. In either event we have

$$p\nu^*(\alpha) - (p-1)^{N+4} \;=\; \frac{1}{(p-1)^{\mathrm{pts}(\Delta)-N-5}} \sum_{\substack{s_{\mathbf{m}}=0 \\ \sum_{\mathbf{m}} s_{\mathbf{m}} \mathbf{m} = \sigma \mathbf{1}}}^{p-2} (-\phi)^{-\sigma} \, G_\sigma \prod_{\mathbf{m}} \alpha_{\mathbf{m}}^{s_{\mathbf{m}}} \, G_{-s_{\mathbf{m}}} \; ,$$

where, in this last expression, $\sigma = \sum_{\mathbf{m}} s_{\mathbf{m}}$. The summation is over all configurations of 'masses' $s_{\mathbf{m}}$ attached to points $\mathbf{m}$ of Δ such that the centroid of the masses is the unique internal point $\mathbf{1}$.

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